

Congestion Games and Coordination mechanisms

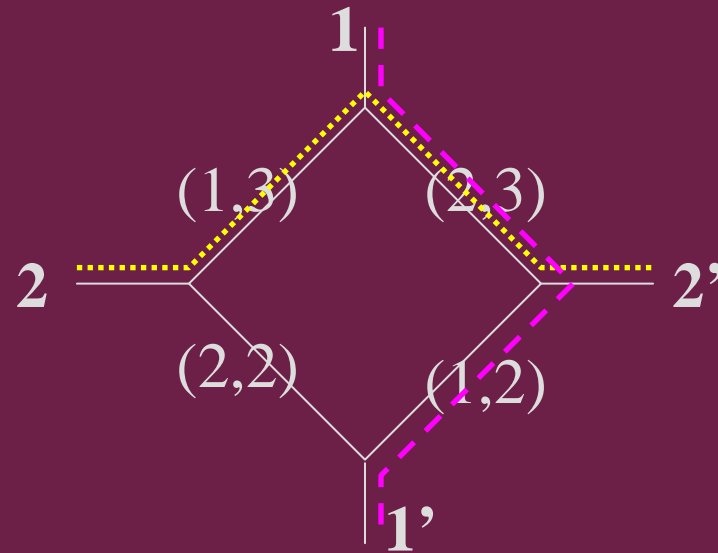
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Outline

- ▶ *Congestion games*
- ▶ *Price of Anarchy*
 - ▶ *General games*
 - ▶ *Linear delay games*
- ▶ *Coping with selfishness*
 - ▶ *Mechanism design*
 - ▶ *k-Implementation*
 - ▶ *Taxes*
- ▶ *Coordination mechanisms*
 - ▶ *2 parallel edges*
 - ▶ *m parallel edges/paths*
 - ▶ *coordination mechanisms for the RT model*



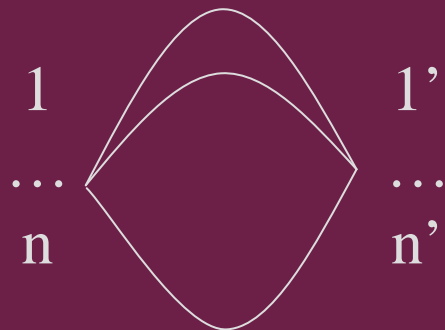
Congestion Games



- ▶ *Each player has a source and a destination. Pure strategies are the paths from source to destination*
- ▶ *The cost on each edge depends on the number of players using the edge: when k players use edge e , each one of the k players incurs cost $c^e(k)$.*

Parallel Links

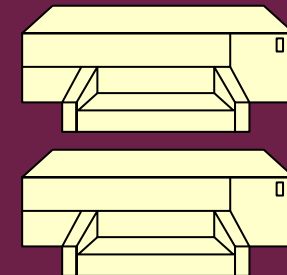
- ▶ *A class of simple congestion games studied in [Koutsoupias, Papadimitriou 1999]: The network consists of m parallel links and the delay on each link is linear (proportional to the number of players using the link).*
- ▶ *Equivalent to the problem of scheduling identical tasks to machines.*



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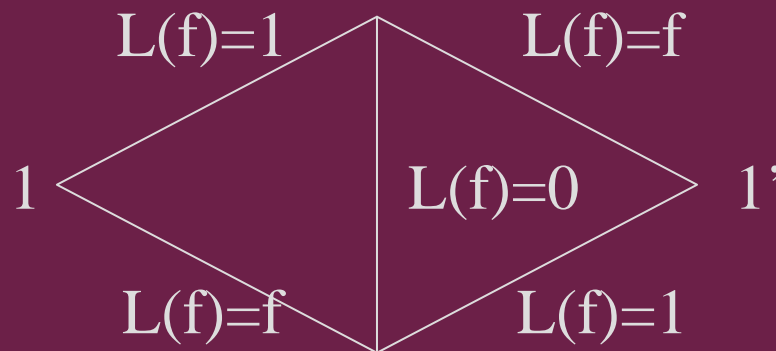
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The Roughgarden-Tardos Model

- ▶ *Roughgarden and Tardos studied another class of congestion games: A continuum of players that use an arbitrary network.*



- ★ *Congestion games is a natural candidate to try to unify the models of Koutsoupias-Papadimitriou and of Roughgarden-Tardos.*
- ★ *They are both extensions of simple congestion games:*
 - ★ *In the Koutsoupias-Papadimitriou model, players have weights: the contributions of the player to the cost of an edge may not be identical.*
 - ★ *In the Roughgarden-Tardos model the number of players is not finite.*

Congestion games - definition

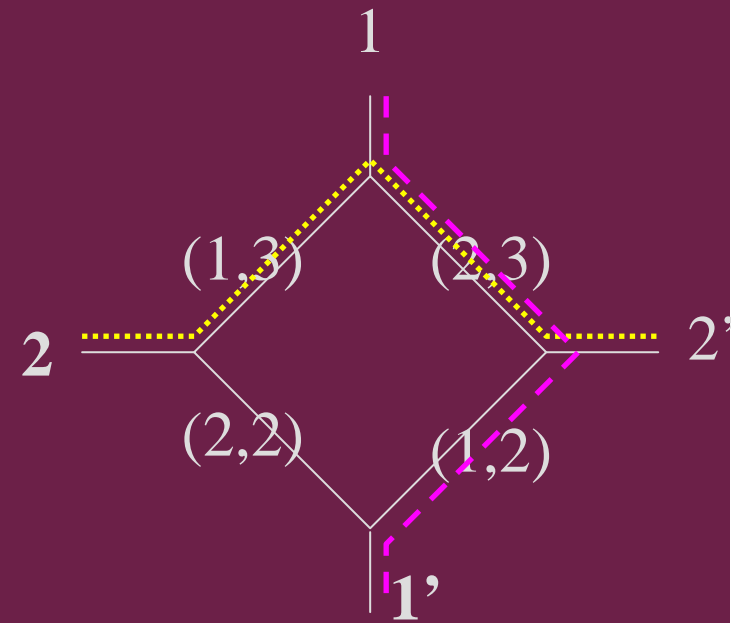
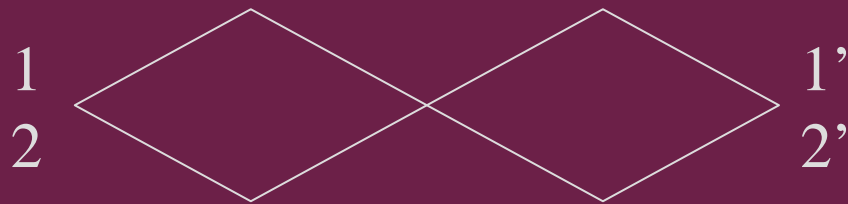
- ▶ n players
- ▶ m facilities (edges)
- ▶ Pure strategy is a subset of facilities – a path. Each player can select from a collection of paths
- ▶ Cost of a facility depends on the number of players using it.
- ▶ The objective of each player is to minimize his own total cost (the sum of the edge costs).

Classes of congestion games

- ▶ *Parallel links - Network - General games*: Is there an underlying network (equivalently, are all paths from source to destination available to each player)?
- ▶ *Symmetric - Asymmetric*: Do players have the same set of strategies (same source and destination)? Also called single-commodity and multi-commodity.
- ▶ *Linear – Arbitrary delay functions*. Is the cost of the facilities proportional to the number of players using it?

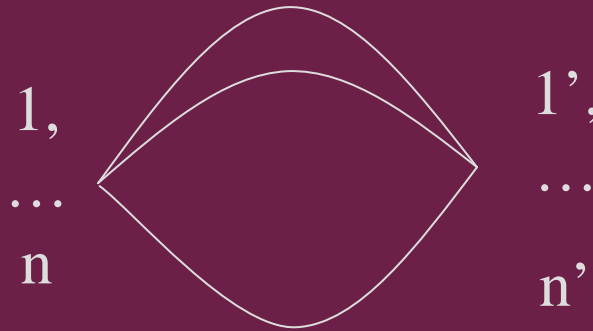
Typical examples – Congestion Games

Simple congestion games



Typical Examples - KP

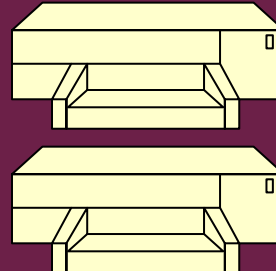
★ *The Koutsoupias-Papadimitriou model*



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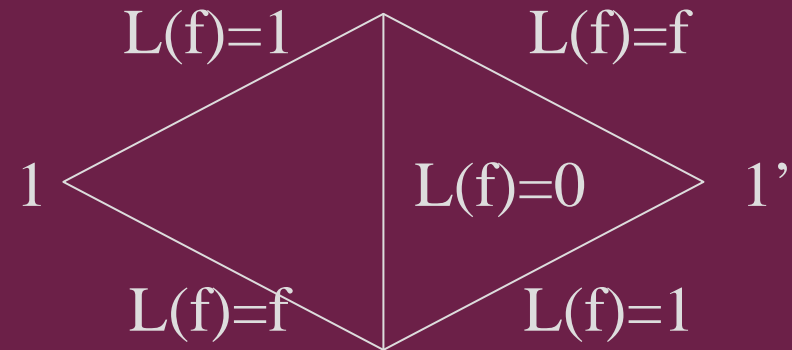
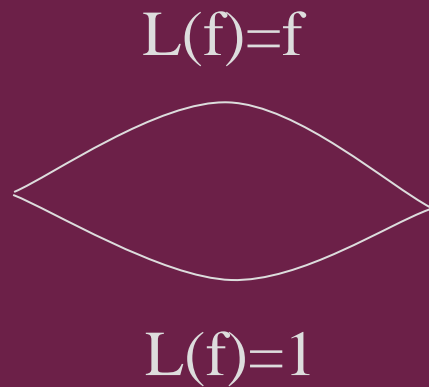
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Typical examples - RT

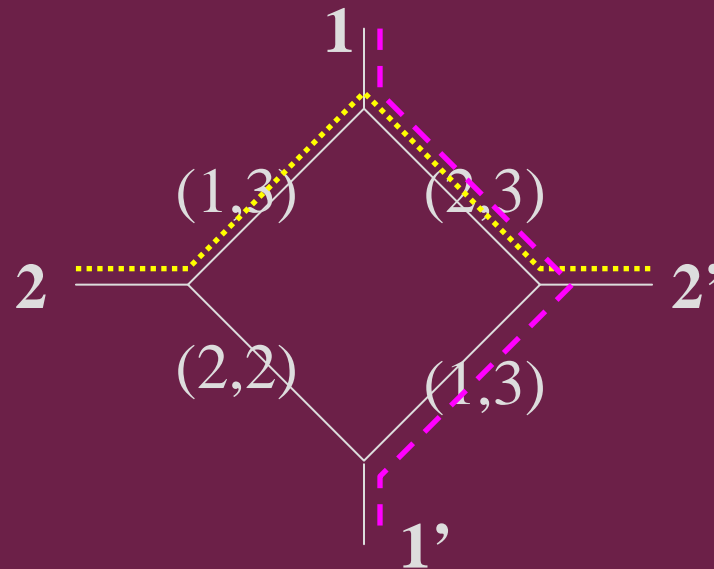
★ *The Roughgarden-Tardos model*



Nash equilibria

- ▶ *Theorem (Rosenthal 1973): Every congestion game has at least one pure Nash equilibrium.*
- ▶ *Proof: Let p_1, \dots, p_n be a collection of pure strategies.*
 - ▶ *The potential P^e of an edge e that appears k times in p_1, \dots, p_n is $\sum_{i=1..k} c^e(i)$.*
 - ▶ *The total potential $P(p_1, \dots, p_n)$ is $\sum_e P^e$.*
 - ▶ *Every local optimum of the potential $P(p_1, \dots, p_n)$ is a Nash equilibrium.*

Example of potential



▲ *Potential* = $1 +$ *from the edge [2,1]*
 $2 + 3 +$ *from edge [1,2']*
 1 *from edge [2',1']*
 $= 7$

History

- ▶ *Rosenthal, 1973*
 - ▶ *Introduced congestion games and showed existence of pure equilibria*
- ▶ *Monterer-Shapley, 1991*
 - ▶ *Showed that they are equivalent to potential games*
- ▶ *Milchtaich, 1996*
 - ▶ *Studied congestion games with player-specific costs.*
- ▶ *Intense recent interest in relation to price of anarchy*
- ▶ *Computational issues: PLS-completeness (Fabrikant, Papadimitriou, Talwar), 2004: **Given a network with costs, it is PLS-complete to compute a pure Nash equilibrium.***



The price of anarchy

- ▶ *When selfish agents share resources, the allocation is usually far from optimal.*
- ▶ *How far from optimal can it be?*
- ▶ *The price of anarchy of a system measures the worst-case selfish cost over the optimal cost:*

$$\text{Price of Anarchy} = \max_{\text{Nash eq } E} \frac{\text{COST}_E}{\text{OPT}}$$

Example

- Consider the following congestion game. The delay on each edge is equal to the number of players using it.



- The first Nash equilibrium is optimal. The costs for the three players are 2,2,2.
- At the second Nash equilibrium, the cost for the players are 2,2,4.
- What is the price of anarchy?

Social cost

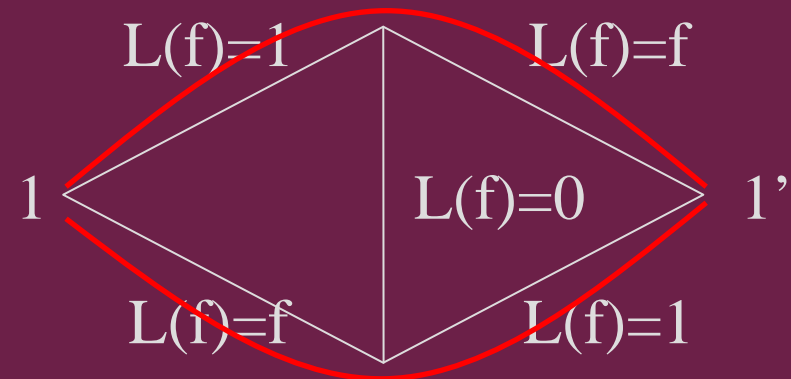
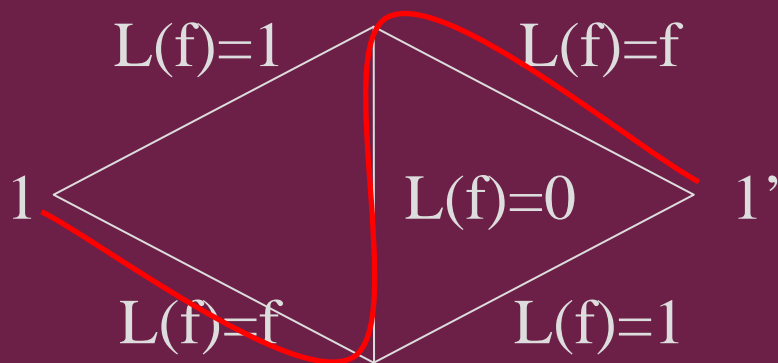
- ▶ *Before we define the price of anarchy, we need to decide what is the social cost.*
- ▶ *There are two obvious possibilities: The average cost or the maximum cost of the players.*
- ▶ *For the **average** cost, the price of anarchy of the example is $(2+2+4)/(2+2+2)=4/3$*
- ▶ *For the **max** cost the price of anarchy is $4/2=2$.*

Price of Anarchy in KP model

- ▶ *For the maximum social cost, the price of anarchy of mixed Nash equilibria is $\log m / \log \log m$
[K-Papadimitriou 99, K-Mavronikolas-Spirakis, Czumaj-Voecking]*
- ▶ *The price of anarchy of pure equilibria is at most 2.*

The price of anarchy in the Roughgarden-Tardos model

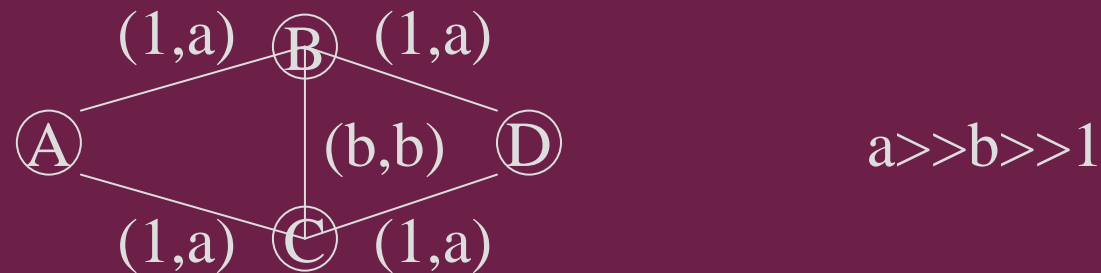
- For $f=1$, the Nash equilibrium on the left has price of anarchy $4/3$ (compare with the optimal solution on the right).



Main problems

- ★ *What is the price of anarchy for the various classes of congestion games?*
- ★ *How can we cope with it?*

Price of anarchy of general congestion games



- ▶ *The price of anarchy of this congestion game is unbounded: $1+b/2$.*
- ▶ *Why? The Nash equilibrium ABCD and ACBD has social cost $2+b$.*
- ▶ *OPT: The strategies ABD and ACD have social cost 2.*

Linear delays, n players

			<i>Parallel links</i>	<i>Network, General</i>
<i>PURE</i>	<i>AVERAGE</i>		<2	$5/2$
	<i>MAX</i>	<i>Symmetric</i>	<2	$5/2$
		<i>Asymmetric</i>	$\log n / \log \log n$	\sqrt{n}
<i>MIXED</i>				??

Christodoulou, Koutsoupias (unpublished)

Gairing, Lucking, Mavronikolas, and Monien (STOC 2004)

Kothari, Suri, Toth and Zhou (unpublished?)

Linear asymmetric games

average cost

▶ Let A_1, \dots, A_n be the strategies/paths at a Nash equilibrium and P_1, \dots, P_n be the optimal strategies.

▶ The cost on each facility is equal to the number of players using it. The cost for player 1 is

$$|A_1| + |A_1 \cap A_2| + \dots + |A_1 \cap A_n|$$

▶ To be at a Nash equilibrium we need to have

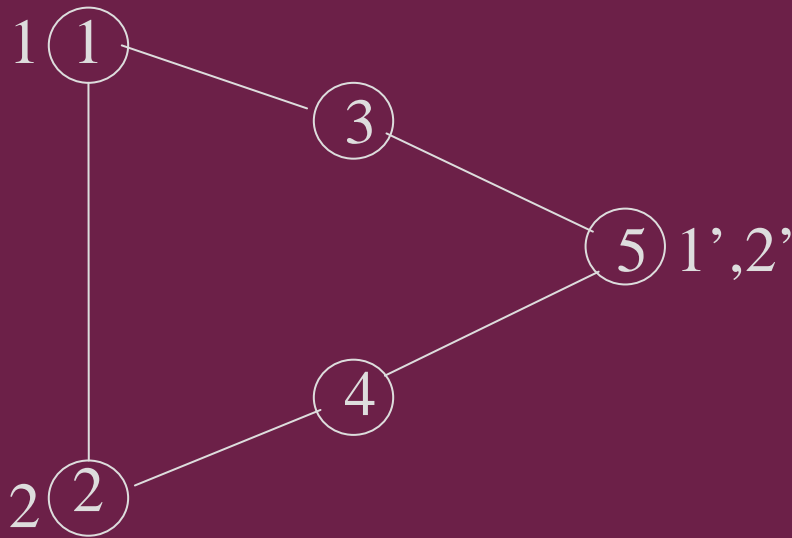
$$|A_1| + |A_2 \cap A_2| + \dots + |A_1 \cap A_n| \leq$$

$$|P_1| + |P_1 \cap A_2| + \dots + |P_1 \cap A_n|$$

and similarly for the other players.

▶ By appropriate set-theoretic transformations we show that the sum of the costs for A_1, \dots, A_n is bounded by $5/2$ times the sum of costs for P_1, \dots, P_n .

Example



$$A_1 = \{e_{12}, e_{24}, e_{45}\}$$

$$A_2 = \{e_{12}, e_{13}, e_{35}\}$$

$$P_1 = \{e_{13}, e_{35}\}$$

$$P_2 = \{e_{24}, e_{45}\}$$

$$\text{cost}_1 = |A_1| + |A_1 \cap A_2| = 4$$

$$\text{opt}_1 = |P_1| + |P_1 \cap P_2| = 2$$

$$\text{Nash: } |A_1| + |A_1 \cap A_2| \leq |P_1| + |P_1 \cap A_2|$$

$$PA = 4/2 = 2$$

▶ *Lemma:*

$$\sum_j (|P_j| + \sum_{i \neq j} |P_j \cap A_i|) \leq \\ 5/3 * \sum_j (|P_j| + \sum_{i \neq j} |P_i \cap P_j|) + \\ 1/3 * \sum_i (|A_i| + \sum_{i \neq j} |A_i \cap A_j|)$$

▶ *Proof:* Consider an element e . Suppose that it belongs to k P_j 's and m A_i 's. Then e contributes to the left side at most $k+km$ and to the right side $5/3 * k^2 + 1/3 * m^2$. Since for nonnegative integer values we have $k+km \leq 5/3 * k^2 + 1/3 * m^2$, the lemma follows.

Extensions of congestion games

- ▶ *Many interesting situations are not exactly congestion games (including the KP and RT models).*
- ▶ *To capture them, we can extend congestion games in two ways:*
 - ▶ *Each player has a **weight** w_i . In this case the cost of the facility depends on the sum of the weights.*
 - ▶ *Miltaich [1996] showed that these games may not have a pure Nash equilibrium.*
 - ▶ *Kontogiannis and Spirakis showed recently that this may be the case even for network congestion games.*
 - ▶ ***Player-specific cost functions** on edges.*



How to cope with anarchy

- ▶ *The traditional approach is mechanism design: payments or tolls improve coordination*
- ▶ *But, it is not feasible for some problems*
 - ▶ *Difficult to compute*
 - ▶ *Extremely high payments*
 - ▶ *Centralized control*
 - ▶ *Others (should drivers bid for crossing an intersection?)*
- ▶ *More importantly, mechanism design focuses on extracting the **truth** from players. Here we deal with **complete information**.*

Taxes

- ▶ *One approach to improve the coordination of players is to introduce taxes: Players pay a toll for every edge they use.*
- ▶ *Cole, Dodis, and Roughgarden showed that there exists taxes that reduce the price of anarchy to 1, for the Roughgarden-Tardos model.*
- ▶ *There are two problems with this approach:*
 - ▶ *Taxes may be very high*
 - ▶ *If taxes are also part of the cost, then the price of anarchy does not improve.*

k-Implementation

- ▶ *The idea is to extend the game by adding new strategies for the players in such a way that*
 - ▶ *all Nash equilibria of the new game use only strategies of the original games*
 - ▶ *The price of anarchy decreases*
- ▶ *Monterer, Tennenholtz 2001*

Coordination mechanisms

- ▶ **Redesign** the game with no (or minimal) additional cost
 - ▶ Slow down some edges (increase their costs)
 - ▶ Introduce player-specific costs.
- ▶ The question is: Which is the **best way** to do it, so that selfish players coordinate?

Games with no weights: coordination mechanisms

★ *Given a congestion game consider all the games G that result when we increase the costs on the edges.*

★ *We want to select the game that has the best price of anarchy:*

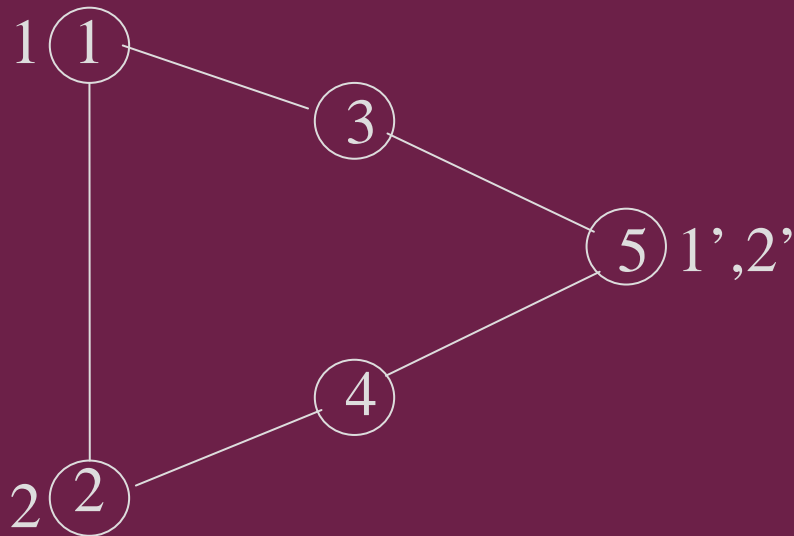
$$\mathbf{\min}_G \mathbf{\max}_{\text{Nash eq } E} \mathbf{cost}(E, G) / \mathbf{opt}$$

★ *We divide with the original optimum, before the increase of costs.*

★ *Does this improve the price of anarchy?*

Example

- ▶ Increase the cost on the edge $[1,2]$ from $c^e(k)=k$ to $c^e(k)=2*k$.
- ▶ The price of anarchy drops from 2 to 1.



Games with no weights: coordination mechanisms

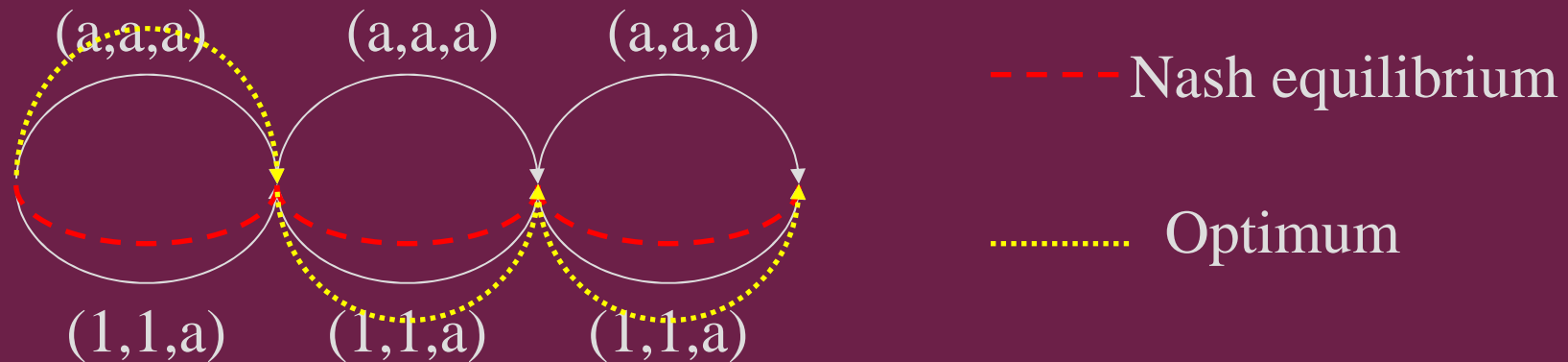
- ★ **Theorem:** *For every **symmetric network** congestion game, there is a way to increase the costs so that the price of anarchy of pure equilibria is n .*
- ★ **Idea for a proof:** *Let P_1, \dots, P_n be a set of paths with optimal (minimum) social cost. If an edge e is used by k paths, increase the cost on the edge when more than k players use it.*

$$\tilde{c}^e(k) = \begin{cases} c^e(i) & i \leq k \\ a & \text{otherwise, where } a \gg 1 \end{cases}$$

Games with no weights: coordination mechanisms

★ *Actually, the proof of the upper bound has a graph-theoretic flavor, uses the fact the game is symmetric, and makes use of the potential of the game.*

★ **Proof of lower bound:**

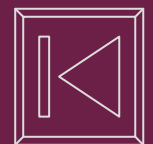


★ **Open problem:** *What is the price of anarchy for **asymmetric** network games and general congestion games?*

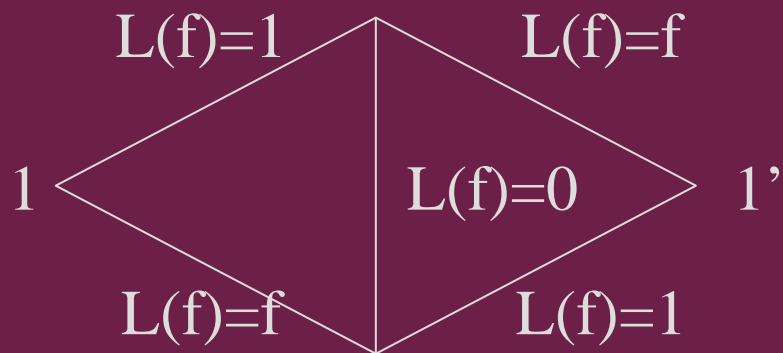
★ *Observation: The mechanism*

$$\tilde{c}^e(k) = \begin{cases} c^e(i) & i \leq k \\ a & \text{otherwise, where } a \gg 1 \end{cases}$$

does not work for general games!



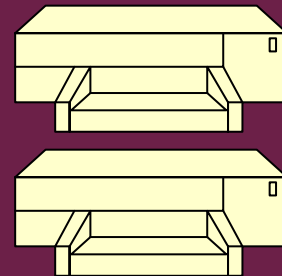
Coordination mechanisms - motivation



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Games with weights: Coordination mechanisms

- ▶ *Consider the case of m parallel edges. The price of anarchy is $\log m / \log \log m$.*
- ▶ *We want to design a coordination mechanism (how to slow down the edges).*
- ▶ *The problem is that we may not know the length of the jobs in advance. The situation has an **online** flavor.*
- ▶ *The interesting question is how to redesign the system to cope with any possible set of jobs.*

Coordination mechanism

- ▶ *Select cost functions c_i^e , one for each edge (player-specific costs that slow down the facility)*
- ▶ *Announce it to players*
- ▶ *When players arrive with weights w_1, \dots, w_n , we compute the worst-case Nash equilibrium.*
- ▶ *The ratio of the cost of this Nash equilibrium over the optimum determines the price of anarchy.*
- ▶ *Question: How to select the cost functions to minimize the price of anarchy?*

The whole framework

Coordination model \leftrightarrow Online problem

Coordination mechanism \leftrightarrow Online algorithm

Price of anarchy \leftrightarrow Competitive Ratio

- ★ *Given a congestion game, we the designers select a coordination mechanism, i.e., cost functions for each facility (essentially a distributed algorithm)*
- ★ *The adversary selects the inputs*
- ★ *We compute the worst-case Nash equilibrium, compare with the optimum and determine the price of anarchy.*

Example: Coordination mechanisms for two parallel edges

- ▶ *Upper edge orders the players in order of increasing weight (breaking the ties between equal-size jobs lexicographically).*
- ▶ *Lower edge orders the players in order of decreasing weight.*
- ▶ *Price of anarchy for 3 players is 1.*
- ▶ *Price of anarchy for 4 players at least 4/3:*
 $(w_1, w_2, w_3, w_4) = (1, 1, 2, 2)$

Mechanism for m identical parallel linear edges

- ▶ *Each edge orders the players in order of decreasing weight.*
- ▶ *Edge e delays each player so that it finishes at the next time t which is equal to $(e \bmod m)$.*
- ▶ *Theorem: Price of anarchy*

$$\frac{4}{3} - \frac{1}{3m}$$

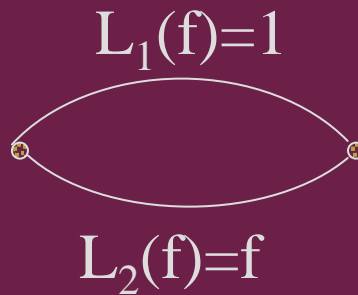
Extensions

- ★ **Theorem:** *For m parallel edges with **arbitrary linear costs**, there exists a coordination mechanism with price of anarchy **$2 - 2/(m+1)$** .*
- ★ *Without a mechanism the price of anarchy is logarithmic.*
- ★ **Open problem:** *What is the optimal coordination mechanism for linear networks?*

Lower bound for 2 edges?

- ▶ *Adversary selects one of the following 4 combinations of player and weights*
 - ▶ $(3, 3, 2, 2, 2)$
 - ▶ $(3, 3, 2, 2, 0)$
 - ▶ $(3, 3, 2, 0, 2)$
 - ▶ $(3, 3, 0, 2, 2)$
- ▶ *In the first case the 3's must go together, in the rest separately.*
- ▶ *Coordination mechanisms cannot distinguish between these cases, because they don't know whether a 2 is missing or is in the other printer.*

Coordination mechanisms for selfish routing

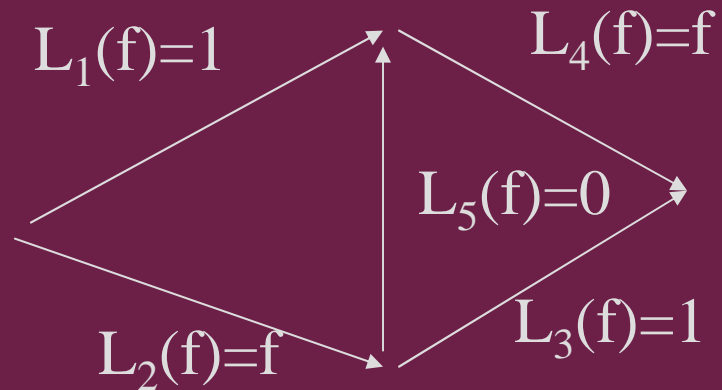


★ *Change $L_2(f)$ to*

$$L_2(f) = \begin{cases} f, & f \leq 1/2 \\ f + 1/2, & f > 1/2 \end{cases}$$

The price of anarchy drops to 1.

Braess' paradox



Change $L_5(f)$ to

$$L_5(f) = \begin{cases} 0, & \text{if } f \leq 2/3 \\ 1, & \text{otherwise} \end{cases}$$

The price of anarchy drops from $4/3$ to $16/15$.

Open Problems

- ★ *What about arbitrary networks and arbitrary delay functions?*
- ★ **Conjecture:** *For the Roughgarden-Tardos model with linear delay functions the price of anarchy improves from $4/3$ to 1.2*