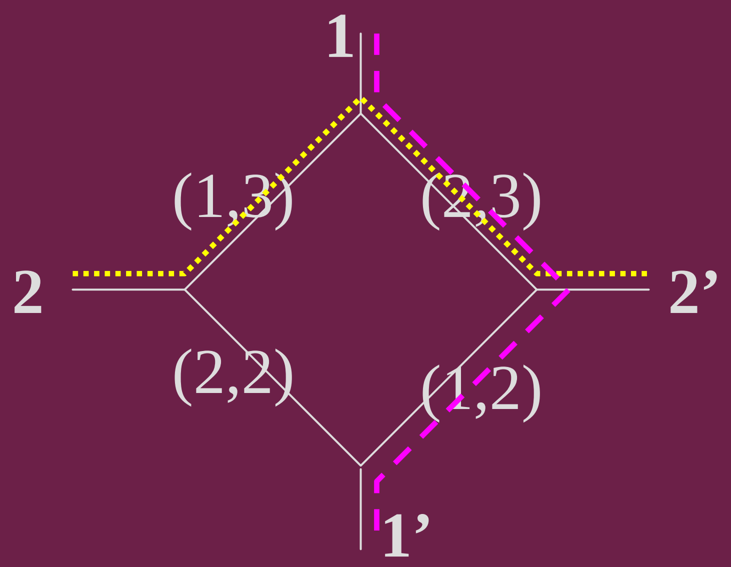
Congestion Games and Coordination mechanisms

Elias Koutsoupias
University of Athens

Outline

- *Congestion games*
- *Price of Anarchy*
  - *General games*
  - *Linear delay games*
- *Coping with selfishness*
  - *Mechanism design*
  - *k-Implementation*
  - *Taxes*
- *Coordination mechanisms*
  - *2 parallel edges*
  - *m parallel edges/paths*
  - *coordination mechanisms for the RT model*

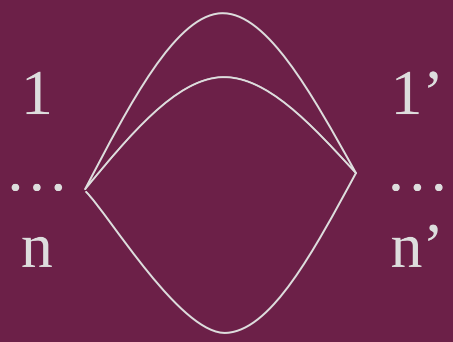
Congestion Games

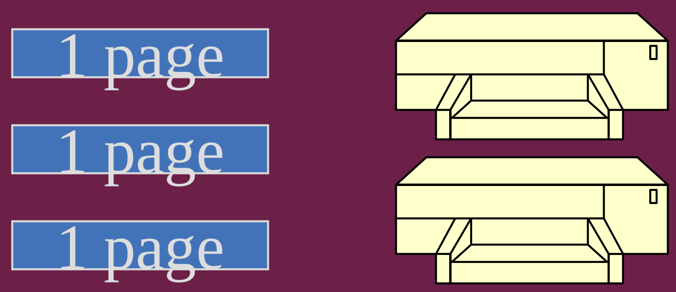


- *Each player has a source and a destination. Pure strategies are the paths from source to destination*
- *The cost on each edge depends on the number of players using the edge: when k players use edge e, each one of the k players incurs cost* $c^e(k)$.

Parallel Links

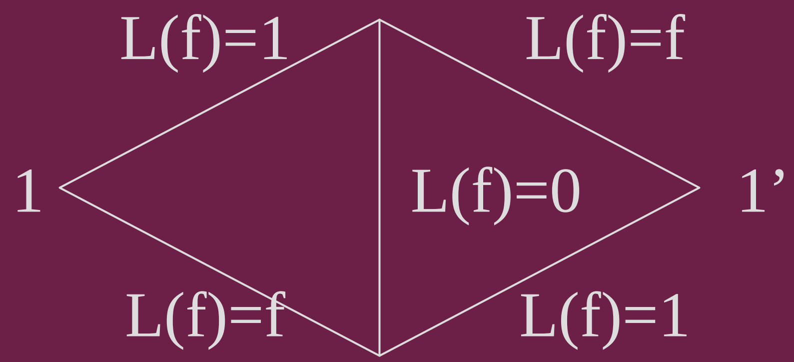
- *A class of simple congestion games studied in [Koutsoupias, Papadimitriou 1999]: The network consists of m parallel links and the delay on each link is linear (proportional to the number of players using the link).*
- *Equivalent to the problem of scheduling identical tasks to machines.*





The Roughgarden-Tardos Model

- *Roughgarden and Tardos studied another class of congestion games: A continuum of players that use an arbitrary network.*



- *Congestion games is a natural candidate to try to unify the models of Koutsoupias-Papadimitriou and of Roughgarden-Tardos.*
- *They are both extensions of simple congestion games:*
  - *In the Koutsoupias-Papadimitriou model, players have weights: the contributions of the player to the cost of an edge may not be identical.*
  - *In the Roughggarden-Tardos model the number of players is not finite.*

Congestion games - definition

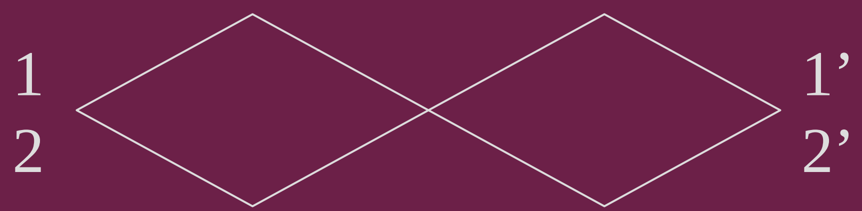
- *n players*
- *m facilities (edges)*
- *Pure strategy is a subset of facilities – a path. Each player can select from a collection of paths*
- *Cost of a facility depends on the number of players using it.*
- *The objective of each player is to minimize his own total cost (the sum of the edge costs).*

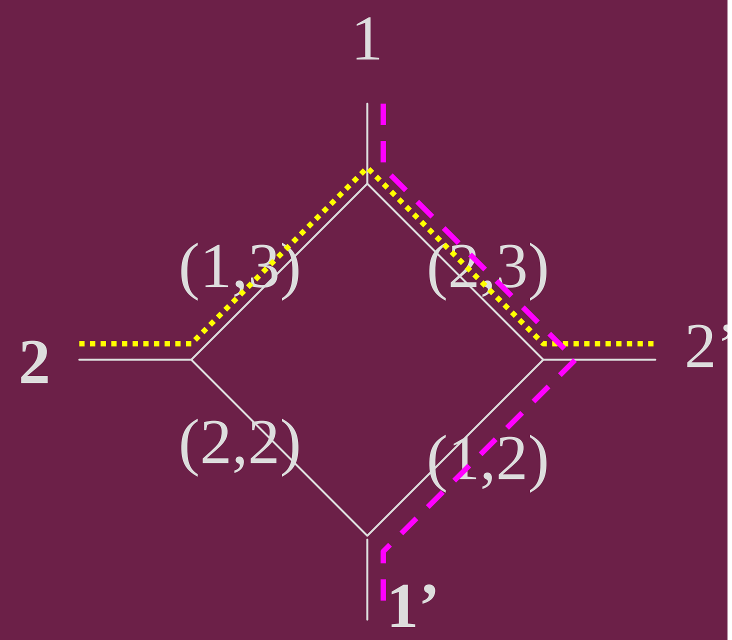
Classes of congestion games

- *Parallel links - Network - General games: Is there an underlying network (equivalently, are all paths from source to destination available to each player)?*
- *Symmetric - Asymmetric: Do players have the same set of strategies (same source and destination)? Also called single-commodity and multi-commodity.*
- *Linear – Arbitrary delay functions. Is the cost of the facilities proportional to the number of players using it?*

Typical examples – Congestion Games

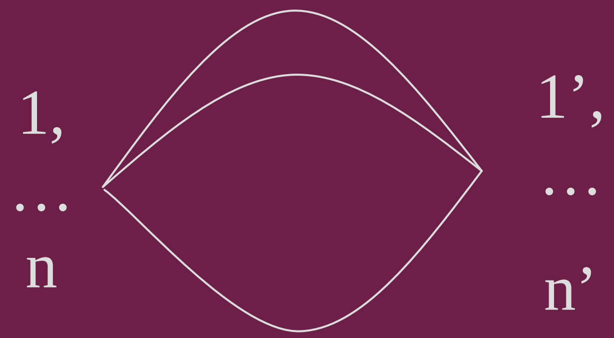
- *Simple congestion games*





Typical Examples - KP

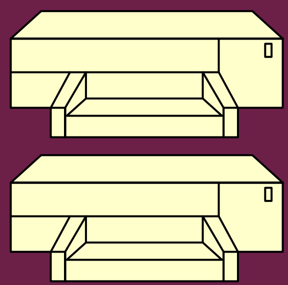
- *The Koutsoupias-Papadimitriou model*



4 pages

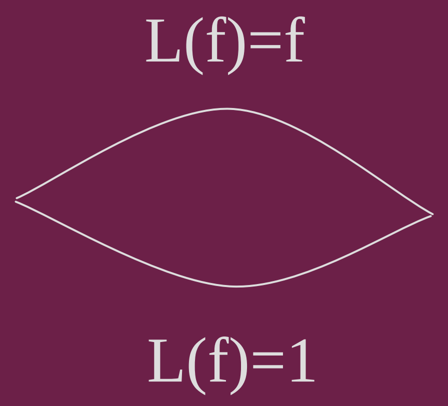
6 pages

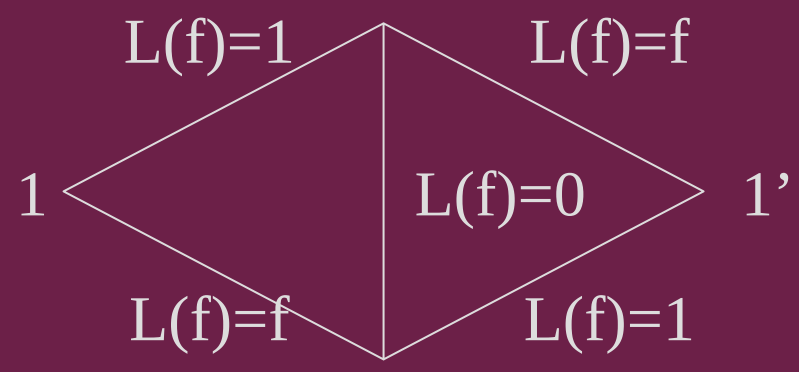
10 pages



Typical examples - RT

- *The Roughgarden-Tardos model*

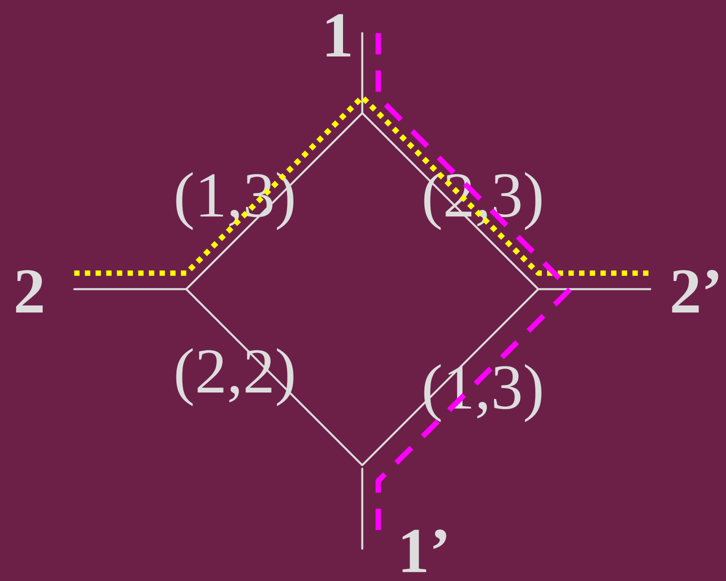




Nash equilibria

- *Theorem (Rosenthal 1973): Every congestion game has at least one pure Nash equilibrium.*
- *Proof: Let $p_1,\ldots,p_n$ be a collection of pure strategies.*
  - *The potential P^e of an edge e that appears k times in $p_1,\ldots,p_n$ is $\sum_{i=1..k} c^e(i)$.*
  - *The total potential $P(p_1,\ldots,p_n)$ is $\sum_e P^e$.*
  - *Every local optimum of the potential $P(p_1,\ldots,p_n)$ is a Nash equilibrium.*

Example of potential



- *Potential =* *1+* *from the edge [2,1]*
  2+3+ *from edge [1,2']*
  1 *from edge [2',1']*
  = 7

History

- *Rosenthal, 1973*
  - *Introduced congestion games and showed existence of pure equilibria*
- *Monterer-Shapley, 1991*
  - *Showed that they are equivalent to potential games*
- *Milchtaich, 1996*
  - *Studied congestion games with player-specific costs.*
- *Intense recent interest in relation to price of anarchy*
- *Computational issues: PLS-completeness (Fabrikant, Papadimitriou, Talwar), 2004:* ***Given a network with costs, it is PLS-complete to compute a pure Nash equilibrium.***

The price of anarchy

- *When selfish agents share resources, the allocation is usually far from optimal.*
- *How far from optimal can it be?*
- *The price of anarchy of a system measures the worst-case selfish cost over the optimal cost:*

$$Price of Anarchy = \max_{\text{Nash eq } E} \frac{\text{COST}_E}{\text{OPT}}$$

Example

- *Consider the following congestion game. The delay on each edge is equal to the number of players using it.*

- *The first Nash equilibrium is optimal. The costs for the three players are 2,2,2.*
- *At the second Nash equilibrium, the cost for the players are 2,2,4.*
- *What is the price of anarchy?*

Social cost

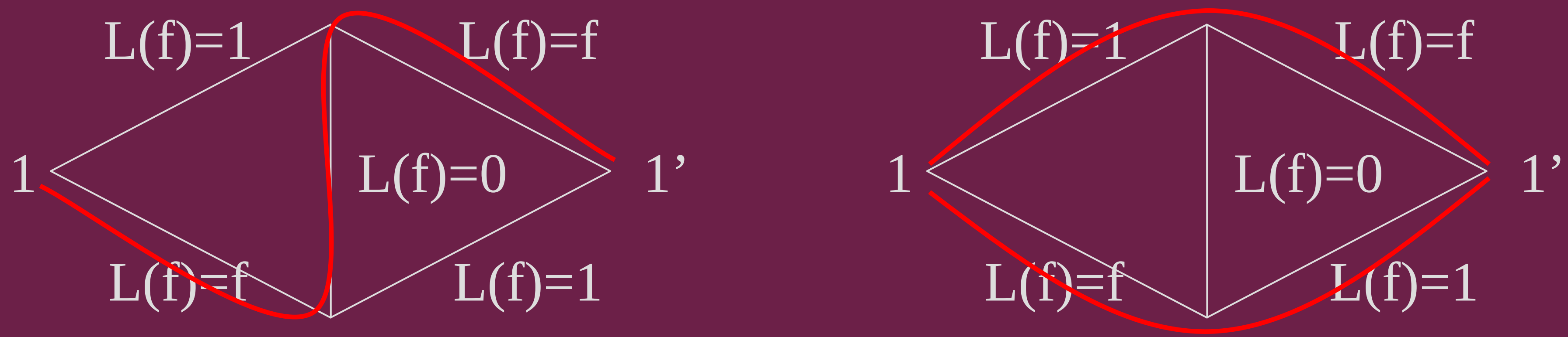
- *Before we define the price of anarchy, we need to decide what is the social cost.*
- *There are two obvious possibilities: The average cost or the maximum cost of the players.*
- *For the* ***average*** *cost, the price of anarchy of the example is (2+2+4)/(2+2+2)=4/3*
- *For the* ***max*** *cost the price of anarchy is 4/2=2.*

Price of Anarchy in KP model

- *For the maximum social cost, he price of anarchy of mixed Nash equilibria is logm/loglogm [K-Papadimitriou 99, K-Mavronikolas-Spirakis, Czumaj-Voecking]*
- *The price of anarchy of pure equilibria is at most 2.*

The price of anarchy in the Roughgarden-Tardos model

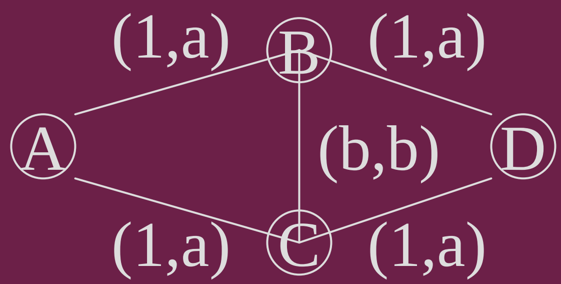
- *For f=1, the Nash equilibrium on the left has price of anarchy 4/3 (compare with the optimal solution on the right).*



Main problems

- *What is the price of anarchy for the various classes of congestion games?*
- *How can we cope with it?*

Price of anarchy of general congestion games



a>>b>>1

- *The price of anarchy of this congestion game is unbounded: 1+b/2.*
- *Why? The Nash equilibrium ABCD and ACBD has social cost 2+b.*
- *OPT: The strategies ABD and ACD have social cost 2.*

Linear delays, n players

			Parallel links	Network, General
PURE	*AVERAGE*		*<2*	*5/2*
	MAX	*Symmetric*	*<2*	*5/2*
		Asymmetric	*logn/loglogn*	$\sqrt{n}$
MIXED			*??*	

Christodoulou, Koutsoupias (unpublished)
Gairing, Lucking, Mavronikolas, and Monien (STOC 2004)
Kothari, Suri, Toth and Zhou (unpublished?)

Linear asymmetric games average cost

- *Let $A_1,\dots,A_n$ be the strategies/paths at a Nash equilibrium and $P_1,\dots,P_n$ be the optimal strategies.*
- *The cost on each facility is equal to the number of players using it. The cost for player 1 is*

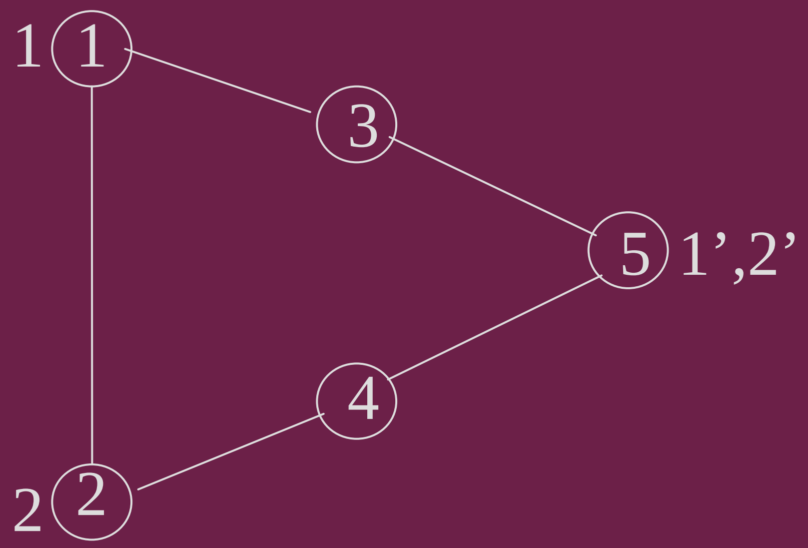
  $$|A_1|+|A_1 \cap A_2|+\dots+|A_1 \cap A_n|$$

- *To be at a Nash equilibrium we need to have*

  $$|A_1|+|A_2 \cap A_2|+\dots+|A_1 \cap A_n| <=$$
  $$|P_1|+|P_1 \cap A_2|+\dots+|P_1 \cap A_n|$$

  and similarly for the other players.
- *By appropriate set-theoretic transformations we show that the sum of the costs for $A_1,\dots,A_n$ is bounded by 5/2 times the sum of costs for $P_1,\dots,P_n$.*

Example



$A_1=\{e_{12},e_{24},e_{45}\}$

$A_2=\{e_{12},e_{13},e_{35}\}$

$P_1=\{e_{13},e_{35}\}$

$P_2=\{e_{24},e_{45}\}$

$cost_1=|A_1|+|A_1\cap A_2|=4$

$opt_1=|P_1|+|P_1\cap P_2|=2$

Nash: $|A_1|+|A_1\cap A_2|\leq |P_1|+|P_1\cap A_2|$

PA=4/2=2

- *Lemma:*

$$\sum_j (|P_j| + \sum_{i\neq j}|P_j \cap A_i|) \leq$$
$$5/3 * \sum_j (|P_j| + \sum_{i\neq j}|P_i \cap P_j|) +$$
$$1/3 * \sum_i (|A_i| + \sum_{i\neq j}|A_i \cap A_j|)$$

- *Proof: Consider an element e. Suppose that it belongs to k P_j's and m A_i's. Then e contributes to the left side at most k+km and to the right side $5/3*k^2+1/3*m^2$. Since for nonnegative integer values we have $k+km \leq 5/3*k^2+1/3*m^2$, the lemma follows.*

Extensions of congestion games

- *Many interesting situations are not exactly congestion games (including the KP and RT models).*
- *To capture them, we can extend congestion games in two ways:*
  - *Each player has a* ***weight*** *w_i. In this case the cost of the facility depends on the sum of the weights.*
    - *Miltaich [1996] showed that these games may not have a pure Nash equilibrium.*
    - *Kontogiannis and Spirakis showed recently that this may be the case even for network congestion games.*
  - ***Player-specific cost functions*** *on edges.*

How to cope with anarchy

- *The traditional approach is mechanism design: payments or tolls improve coordination*
- *But, it is not feasible for some problems*
  - *Difficult to compute*
  - *Extremely high payments*
  - *Centralized control*
  - *Others (should drivers bid for crossing an intersection?)*
- *More importantly, mechanism design focuses on extracting the **truth** from players. Here we deal with **complete information**.*

Taxes

- *One approach to improve the coordination of players is to introduce taxes: Players pay a toll for every edge they use.*
- *Cole, Dodis, and Roughgarden showed that there exists taxes that reduce the price of anarchy to 1, for the Roughgarden-Tardos model.*
- *There are two problems with this approach:*
  - *Taxes may be very high*
  - *If taxes are also part of the cost, then the price of anarchy does not improve.*

k-Implementation

- *The idea is to extend the game by adding new strategies for the players in such a way that*
  - *all Nash equilibria of the new game use only strategies of the original games*
  - *The price of anarchy decreases*
- *Monterer, Tennenholtz 2001*

Coordination mechanisms

- ***Redesign*** *the game with no (or minimal) additional cost*
  - *Slow down some edges (increase their costs)*
  - *Introduce player-specific costs.*
- *The question is: Which is the* ***best way*** *to do it, so that selfish players coordinate?*

Games with no weights: coordination mechanisms

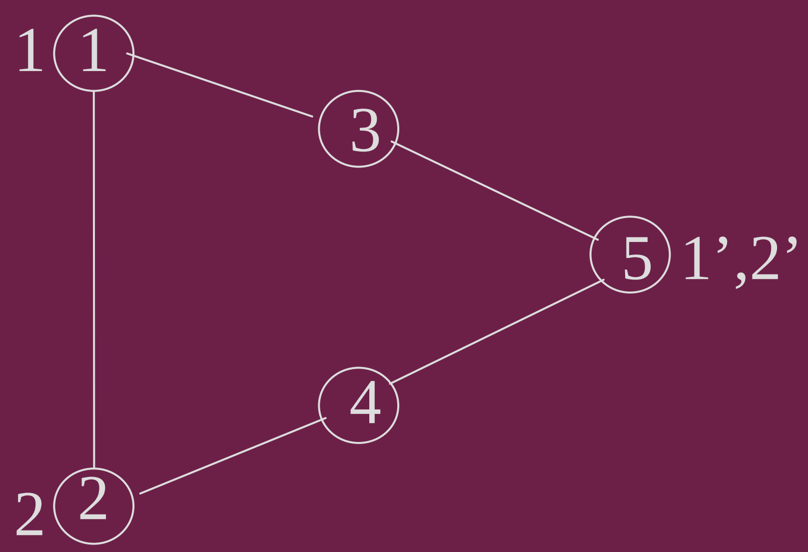
- *Given a congestion game consider all the games G that result when we increase the costs on the edges.*
- *We want to select the game that has the best price of anarchy:*

$$\min_G \max_{Nash\ eq\ E} cost(E, G) / opt$$

- *We divide with the original optimum, before the increase of costs.*
- *Does this improve the price of anarchy?*

Example

- *Increase the cost on the edge [1,2] from c^e(k)=k to c^e(k)=2*k.*
- *The price of anarchy drops from 2 to 1.*



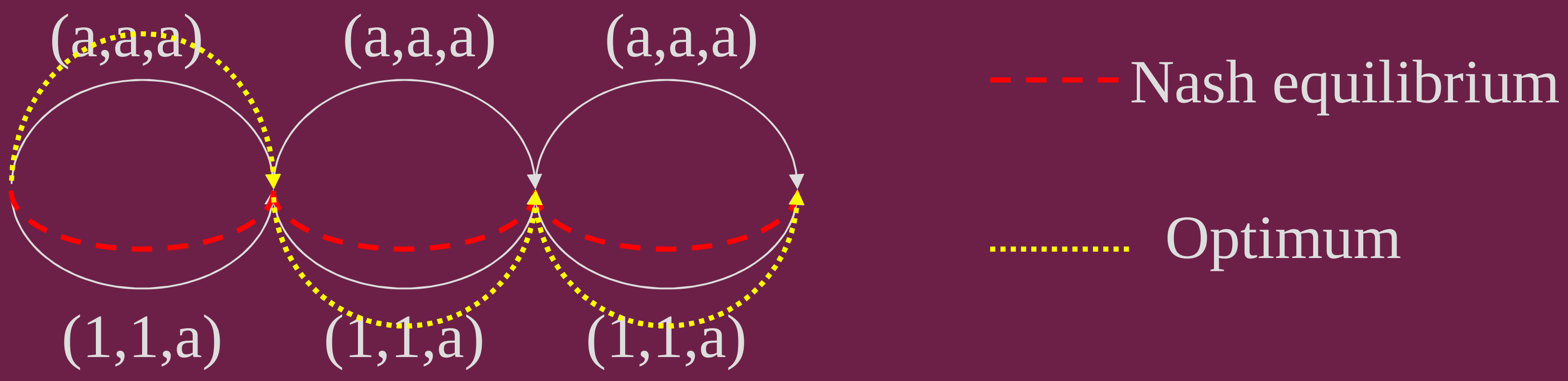
Games with no weights: coordination mechanisms

- **Theorem:** *For every **symmetric network** congestion game, there is a way to increase the costs so that the price of anarchy of pure equilibria is n.*
- **Idea for a proof:** *Let $P_1,\ldots,P_n$ be a set of paths with optimal (minimum) social cost. If an edge e is used by k paths, increase the cost on the edge when more than k players use it.*

$$\tilde{c}^e(k) = \begin{cases} c^e(i) & i \leq k \\ a & \text{otherwise, where } a >> 1 \end{cases}$$

Games with no weights: coordination mechanisms

- *Actually, the proof of the upper bound has a graph-theoretic flavor, uses the fact the game is symmetric, and makes use of the potential of the game.*
- **Proof of lower bound**:

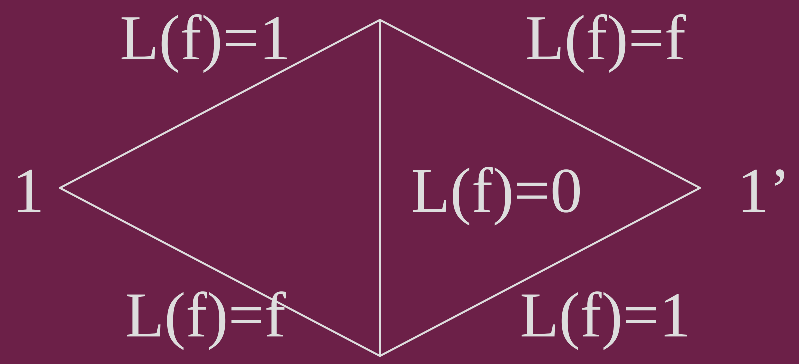


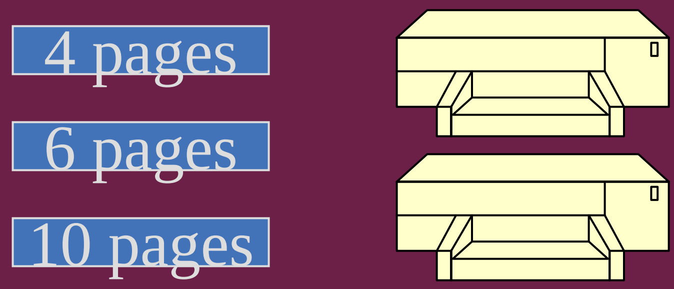
- **Open problem:** *What is the price of anarchy for* ***asymmetric*** *network games and general congestion games?*
- *Observation: The mechanism*

$$\tilde{c}^e(k) = \begin{cases} c^e(i) & i \leq k \\ a & \text{otherwise, where } a >> 1 \end{cases}$$

does not work for general games!

Coordination mechanisms - motivation





Games with weights: Coordination mechanisms

- *Consider the case of m parallel edges. The price of anarchy is logm/loglogm.*
- *We want to design a coordination mechanism (how to slow down the edges).*
- *The problem is that we may not know the length of the jobs in advance. The situation has an **online** flavor.*
- *The interesting question is how to redesign the system to cope with any possible set of jobs.*

Coordination mechanism

- *Select cost functions* c_i^e*, one for each edge (player-specific costs that slow down the facility)*
- *Announce it to players*
- *When players arrive with weights* $w_1, \ldots, w_n$*, we compute the worst-case Nash equilibrium.*
- *The ratio of the cost of this Nash equilibrium over the optimum determines the price of anarchy.*
- *Question: How to select the cost functions to minimize the price of anarchy?*

The whole framework

Coordination model ←→ Online problem

Coordination mechanism ←→ Online algorithm

Price of anarchy ←→ Competitive Ratio

- *Given a congestion game, we the designers select a coordination mechanism, i.e., cost functions for each facility (essentially a distributed algorithm)*
- *The adversary selects the inputs*
- *We compute the worst-case Nash equilibrium, compare with the optimum and determine the price of anarchy.*

Example: Coordination mechanisms for two parallel edges

- ***Upper edge*** *orders the players in order of* ***increasing*** *weight (breaking the ties between equal-size jobs lexicographically).*
- ***Lower edge*** *orders the players in order of* ***decreasing*** *weight.*
- *Price of anarchy for 3 players is 1.*
- *Price of anarchy for 4 players at least 4/3:*

$(w_1,w_2,w_3,w_4)=(1,1,2,2)$

Mechanism for m identical parallel linear edges

- *Each edge orders the players in order of decreasing weight.*
- *Edge e delays each player so that it finishes at the next time t which is equal to (e mod m).*
- *Theorem: Price of anarchy*

$$\frac{4}{3} - \frac{1}{3m}$$

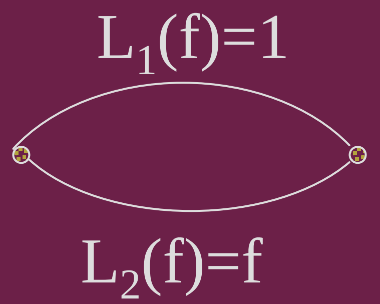
Extensions

- **Theorem:** *For m parallel edges with **arbitrary linear costs**, there exists a coordination mechanism with price of anarchy **2-2/(m+1)**.*
- *Without a mechanism the price of anarchy is logarithmic.*
- **Open problem:** *What is the optimal coordination mechanism for linear networks?*

Lower bound for 2 edges?

- *Adversary selects one of the following 4 combinations of player and weights*
  - *(3, 3, 2, 2, 2)*
  - *(3, 3, 2, 2,* ***0****)*
  - *(3, 3, 2,* ***0****, 2)*
  - *(3, 3,* ***0****, 2, 2)*
- *In the first case the 3's must go together, in the rest separately.*
- *Coordination mechanisms cannot distinguish between these cases, because they don't know whether a 2 is missing or is in the other printer.*

Coordination mechanisms for selfish routing

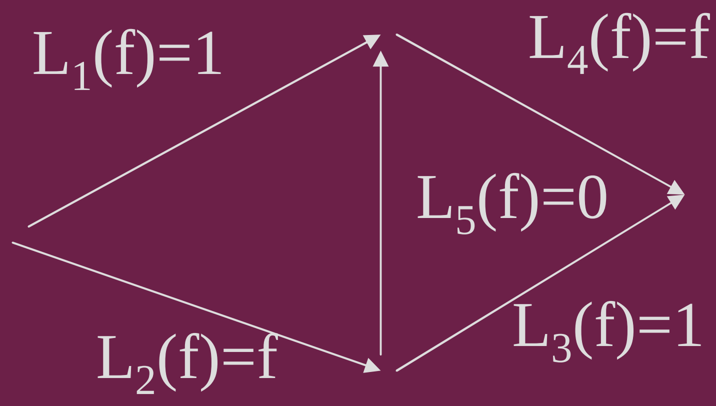


- *Change $L_2(f)$ to*

$$L_2(f) = \begin{cases} f, & f \leq 1/2 \\ f + 1/2, & f > 1/2 \end{cases}$$

The price of anarchy drops to 1.

Braess' paradox



Change $L_5(f)$ to

$$L_5(f) = \begin{cases} 0, \text{if } f \leq 2/3 \\ 1, \text{otherwise} \end{cases}$$

The price of anarchy drops from 4/3 to 16/15.

Open Problems

- *What about arbitrary networks and arbitrary delay functions?*
- **Conjecture:** *For the Roughgarden-Tardos model with linear delay functions the price of anarchy improves from 4/3 to 1.2*