Summary of Lecture 1

The simple case

Lecture 2: The complicated case

Statistical Mechanics of systems of heterogeneous interacting agents

Matteo Marsili
Abdus Salam International Centre for Theoretical Physics

- Key features:
  - many (∞) agents/resources
  - heterogeneity
  - bounded rationality
- Example: El Farol bar problem
- Fluctuations: the Minority Game
- The stationary state and Nash Equilibria
  → the price of naïveness
- Learning to coordinate on Nash Equilibria

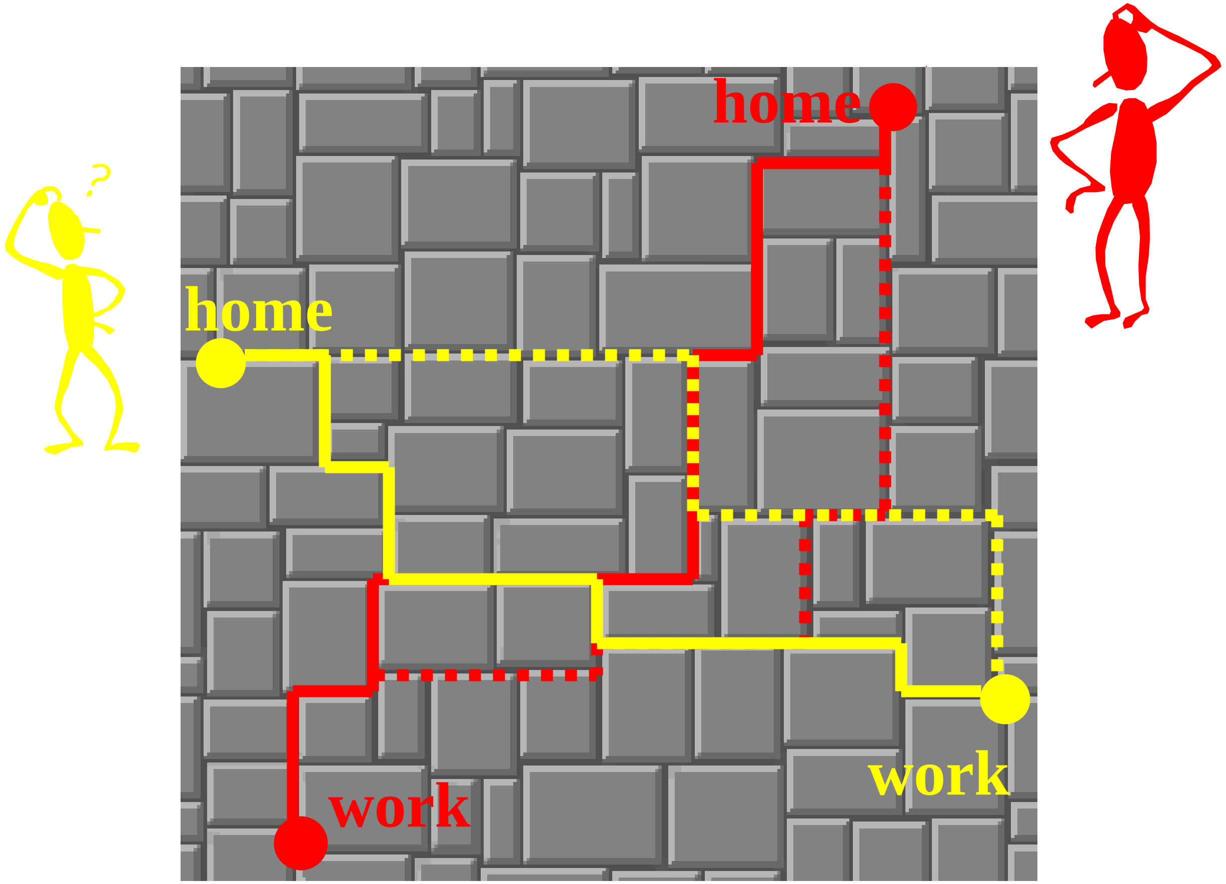
Forthcoming books

- The Minority Game
  D. Challet, M. Marsili, Y.-C. Zhang
  (Oxford U. Press Oct. 2004)
- Statistical Mechanics of Minority Games
  A.A.C. Coolen
  (Oxford U. Press Feb. 2005)

- Web page on Minority Game (google)

D. Challet (Oxford), Y.-C. Zhang, G. Ottino (Fribourg CH),
J. Berg, F. Ricci-Tersenghi, R. Mulet, R. Zecchina … (ICTP Trieste),
A. Rustichini (Minnesota), A. Chessa (Cagliari)
A. De Martino, I. Giardina, A Tedeschi (Roma), M. Piai (SISSA) …
… D. Sherrington, T. Galla (Oxford), A. Cavagna (Roma),
R. Savit (U. Mich.), W.B. Arthur (Santa-Fe) …

A typical problem:



Agents, players = drivers, surfers, buyers, speculators,…

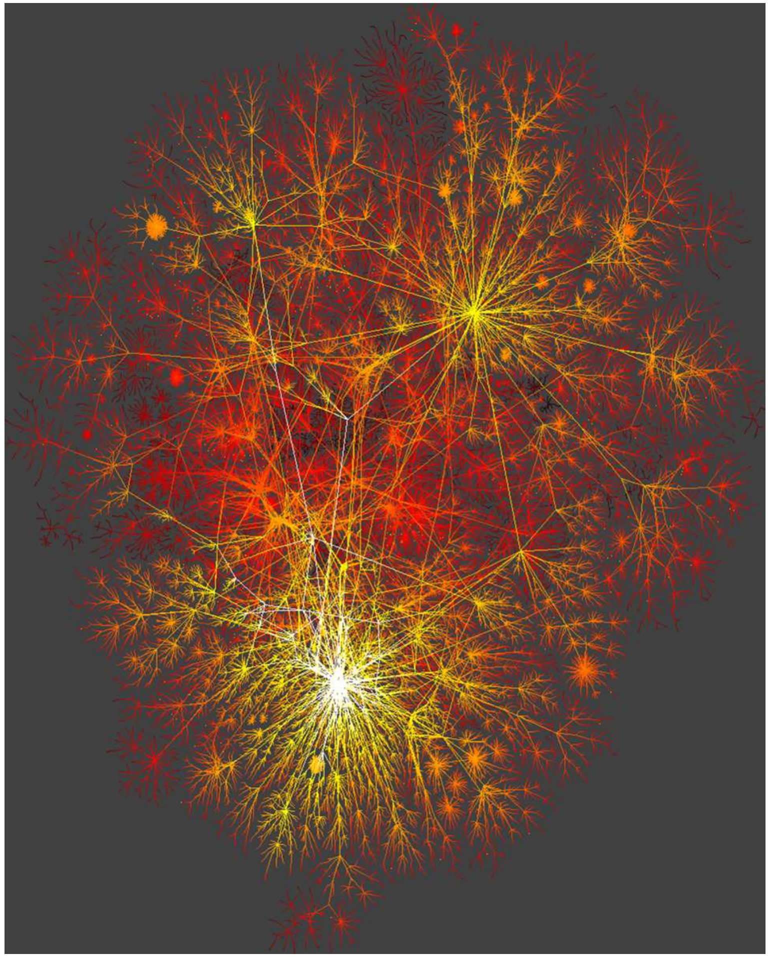
Resources, facilities = streets, channels, sellers, arbitrages,…

$$\text{time(path)} \propto \#\text{of agents along path}$$

$$\text{price(seller)} \propto \#\text{of buyers}$$

In many realistic cases:

- # players ~ # nodes ~ $10^6 \div 10^8$
  ~ ∞ size systems
- Heterogeneous players/nodes
  → random "games"
  Typical behavior, fluctuations
- # strategies ~ $2^{\#\text{ links}}$
  game theory is very demanding!
  → bounded rationality
  - reduced strategy set
  - adaptive learning
  - price taking behavior
  - finite memory…



A class of complex adaptive systems

- N agents
  - •adaptive, low rationality (few behavioral rules)
  - •heterogeneous (quenched disorder)
- P resources
  - •distributed (in space or time)
  - •depleted by use (competition)
- $N \to \infty, \quad P \to \infty, \quad P/N \to \alpha$ finite

```
E.g. financial markets (how does information flows into prices),
     street markets (how do goods flow from sellers to buyers),
     production economies (how do raw resources flow into goods),
     urban traffic (how do cars flow across streets),
     internet traffic (how does information flows through the net),
     ecosystems species/resources,
     ...
```

The “thermodynamic” limit

- $N \to \infty, \quad P \to \infty, \quad P/N \to \alpha \quad \text{finite}$

For N finite P$\to\infty$ (α=0) or P finite N$\to\infty$ (α= ∞) the collective behavior can be understood by relatively simple arguments

Complexity (=non-trivial collective behavior) arises for intermediate values of α

Heterogeneity and random games

Focus on self-averaging quantities (i.e. which satisfy laws of large numbers).

Typical results for N $\rightarrow \infty$

$$P\left(|\,\text{sample} - \text{typical}\,| > \varepsilon\right) \xrightarrow[N\to\infty]{} 0$$

(as opposed to worst case analysis)

Bounded rationality

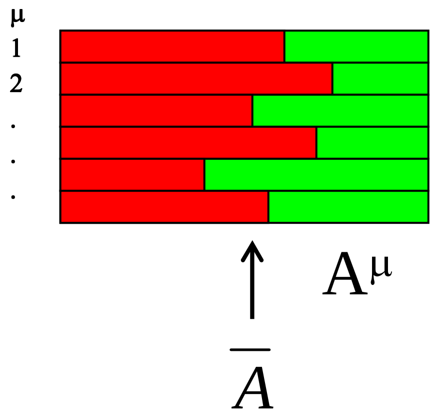
- ***Behavioral rules (schemata):***
  only a finite subset of S pure strategies (among the 2^P possible ones) are considered by agents
  (finite computational cost as P $\to \infty$)
- ***Adaptive learning:***
  - the worth of strategies is determined by how they fare against those played by other agents.
  - a strategies is played more frequently the more it is successful
- ***Price taking behavior:***
  agents play *as if* playing against "the market" rather than against N-1 players
  e.g. price of oranges is the same whether I buy oranges or apples. Indeed

$$\frac{\partial p(\text{orange})}{\partial \,\text{my demand}} \propto 1/N \approx 0$$

so it seems harmless assumption but it simplifies things a lot.

Questions?

$A^\mu = \Sigma_i a_i^\mu$ load of resource $\mu = 1, \ldots, P$



- Where does average load $\overline{A}$ converges?
- How evenly are resources exploited?

$$H = \frac{1}{P}\sum_{\mu=1}^{P}\left(\left\langle A^\mu \right\rangle - \overline{A}\right)^2, \qquad \overline{A} = \frac{1}{P}\sum_{\mu=1}^{P}\left\langle A^\mu \right\rangle$$

- How large are fluctuations of resource loads?

$$\sigma^2 = \frac{1}{P}\sum_{\mu=1}^{P}\left\langle \delta A^\mu \right\rangle^2, \qquad \left\langle \delta A^\mu \right\rangle = A^\mu - \left\langle A^\mu \right\rangle$$

- How does collective behavior depends on agents' behavior?
- How/when do agents learn to coordinate?
- ...

Example: The El Farol bar problem

… *N* people decide independently each week whether to go to a bar that offers entertainment on a certain night. For concreteness, let us set *N* Êat 100. Space is limited, and the evening is enjoyable if things are not too crowded—specifically, if fewer than 60% of the possible 100 are present. There is no way to tell the numbers coming for sure in advance, therefore a person or agent: *goe*s—deems it worth going—if he expects fewer than 60 to show up, or *stays home* if he expects more than 60 to go. Choices are unaffected by previous visits; there is no collusion or prior communication among the agents; and the only information available is the numbers who came in past weeks… (Arthur 1994)

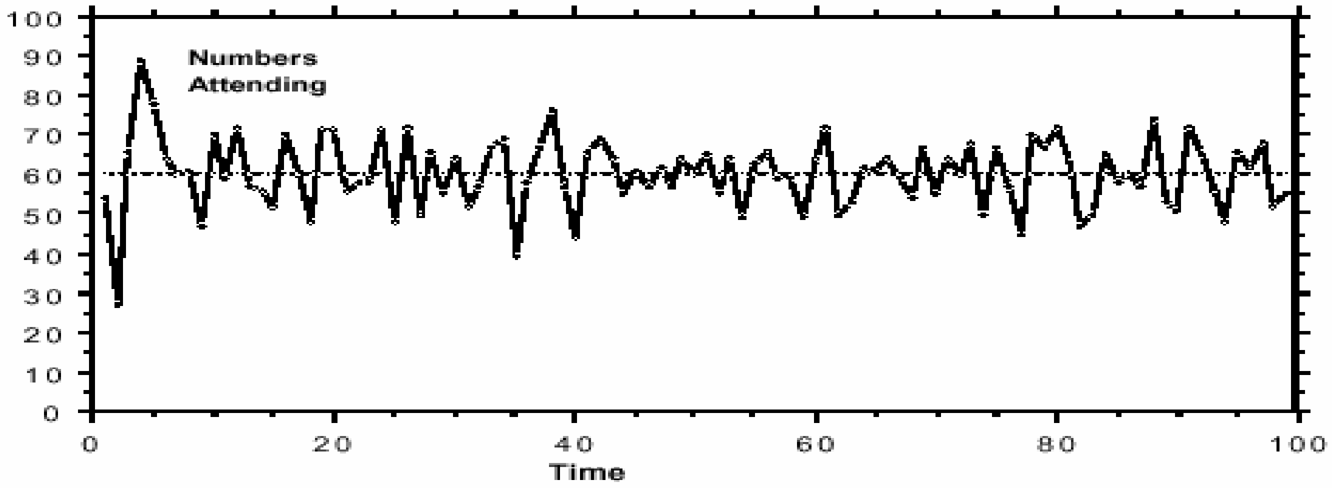


FIGURE 1. BAR ATTENDANCE IN THE FIRST 100 WEEKS.

Schemata in El Farol bar

- A(t)=attendance at week t
- Information

  🖐$_t$={A(t-1),A(t-2),…,A(t-m)}

  resources = bar capacity given information 🖐
- Each agent i has S predictors (randomly drawn)

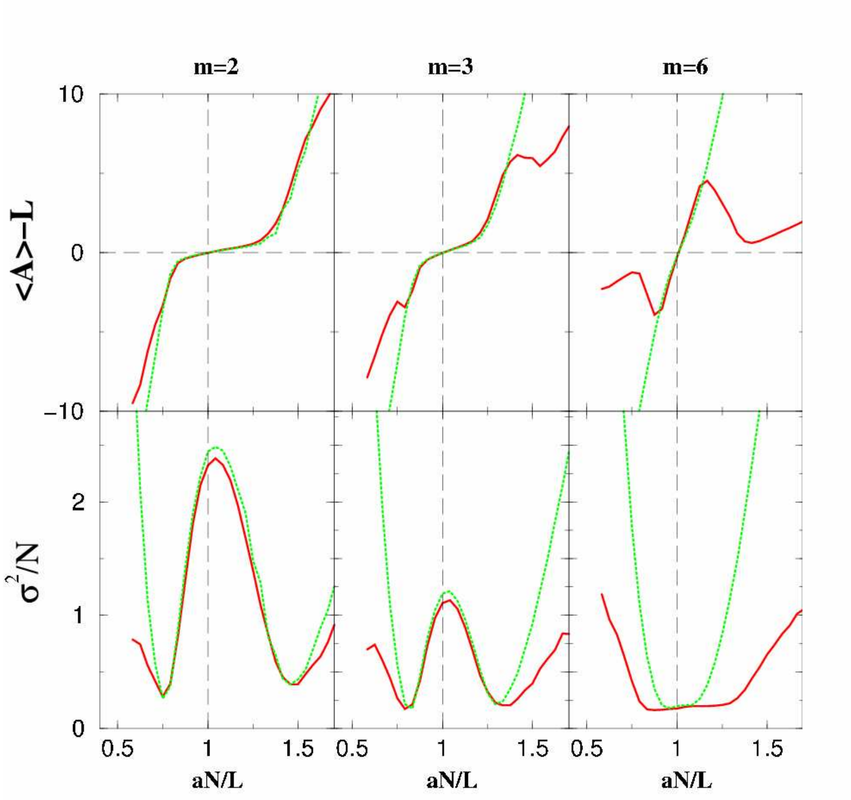
  ✌$_{i,s}$: 🖐→ ✌$_{i,s}$(🖐),   s=1,…,S
- Agents follow the best predictor s*

  if ✌$_{i,s*}$(🖐$_♦$) < 60 then go

  else stay home

Convergence to comfort level L is trivial in El Farol bar

- If predictors are unbiased $P\{✌_{i,s}(✋) < x\,N\} = x$
  $\rightarrow E[A(t)] = N\,P\{✌_{i,s}(✋) < 60\} = 0.6\,N = L$



predictor's bias

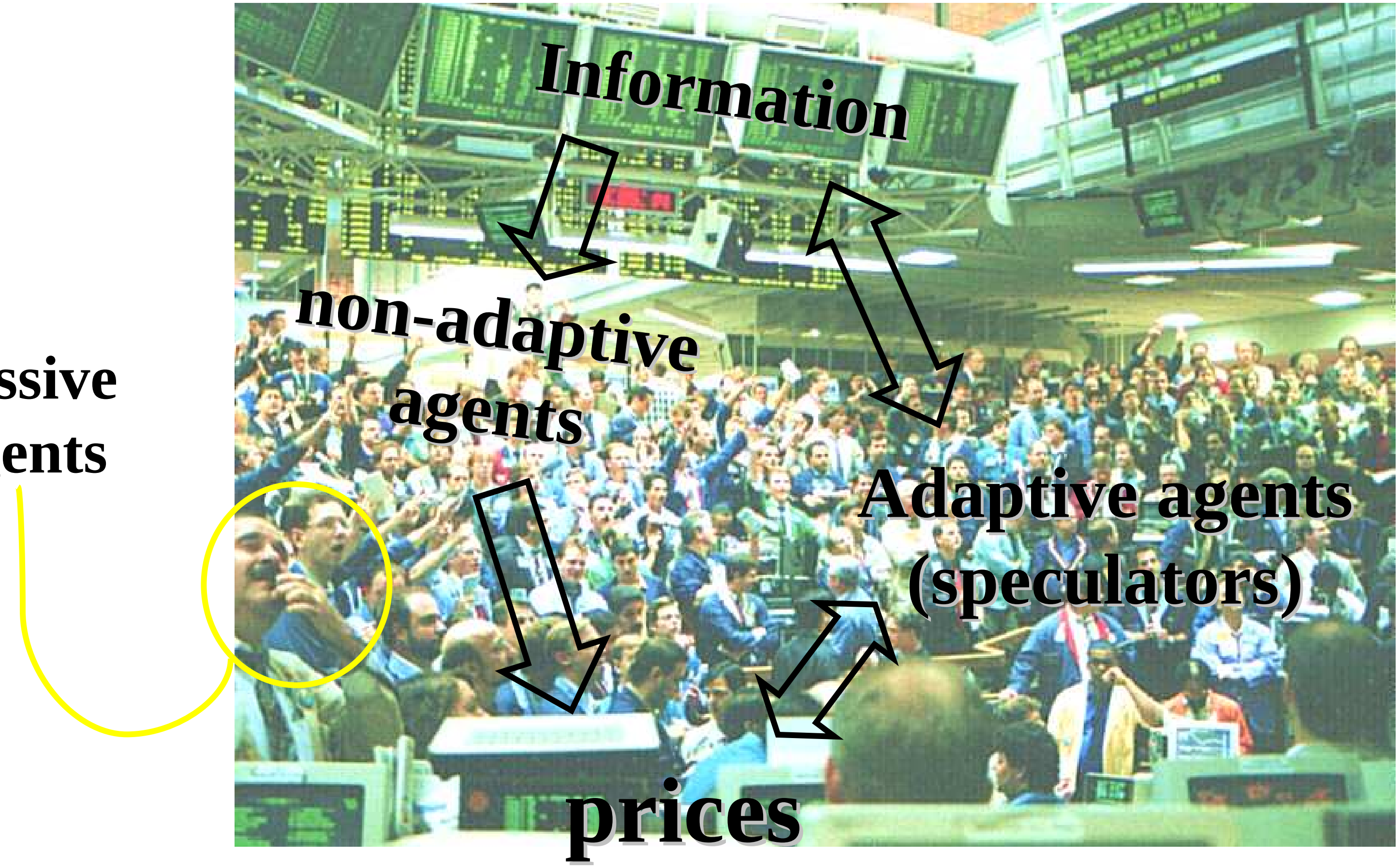
- Fluctuations non-trivial

  $\sigma^2=E[(A-L)^2]$

  fluctuations ~ social cost
- $\rightarrow$ Minority Game

Challet, MM, Ottino PRE 2003

The Minority Game: a stylized model of a financial market

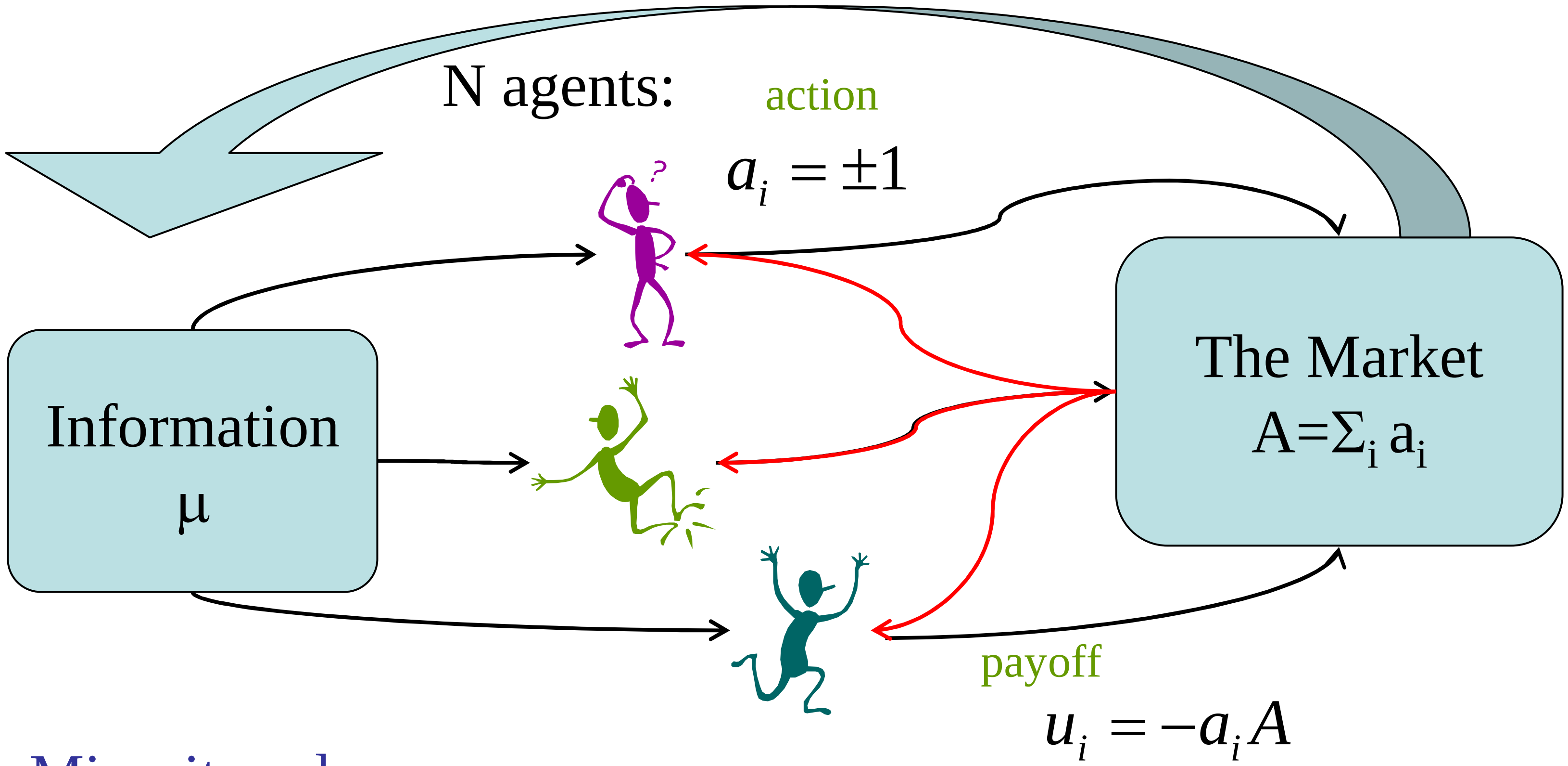


Agents interact through information (mean field interaction) which is how information is incorporated into prices.

Details: agents trade on different time horizons,
they have different weights (capitals) which evolve,
they may refrain from trading,
there are rare and less rare events…

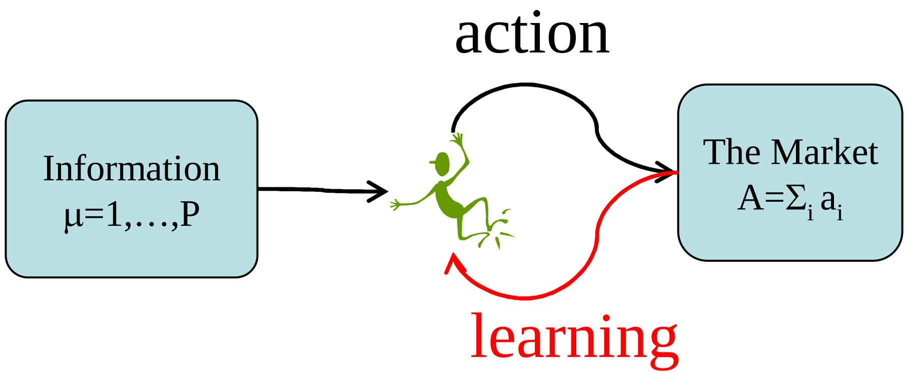
The minority game:

(Challet & Zhang 1997)



- Minority rule
- Agents = set of trading strategies $\{a_i: \mu \rightarrow a_i^\mu\}$ +
  learning and adaptation
- Agents are different: different strategies (quenched disorder)
- Information μ endogenous or exogenous (=random in {1,…,P})

Adaptive agents:



- **Behavioral rules:** $a^{\mu}_{s,i}, \quad s = \pm 1$

  gives action a_i depending on information μ (for all s)
  information processing device/forecasting rule
  randomly drawn at t=0, before the game starts

- **Learning:**

  Scores $U_{s,i}(t)$

$$U_{s,i}(t+1) = U_{s,i}(t) - a^{\mu(t)}_{s,i} \frac{A(t)}{N}, \quad A(t) = \sum_{j=1}^{N} a^{\mu(t)}_{s_j(t),j}$$

- **Choice:**

$$P\{s_i(t) = s\} \propto e^{\Gamma U_{s,i}(t)}$$

Minority game with players

- μ is chosen by Nature $P\{\mu=\nu\}=1/P$
- s_i is chosen by players so as to maximize

$$u_i(s_i, s_{-i}) = -\frac{1}{P}\sum_{\mu=1}^{P}\sum_{j=1}^{N} a_{i,s_i}^{\mu} a_{j,s_j}^{\mu} = NE_{j,\mu}\left[a_{i,s_i}^{\mu} a_{j,s_j}^{\mu}\right]$$

(e.g. as if players were randomly matched)

- Mixed strategies: $\pi_{i,s}=P\{s_i=s\}$

- Is the stationary state of the Minority Game (close to) a Nash Equilibrium?

The answer is

not

- *the stationary state of the Minority Game is related to min H*
- *the Nash equilibria are related to min* σ^2

The stationary state
of the Minority Game:

Numerical results:

- scaling α=P/N

(Savit et al PRL 1998)

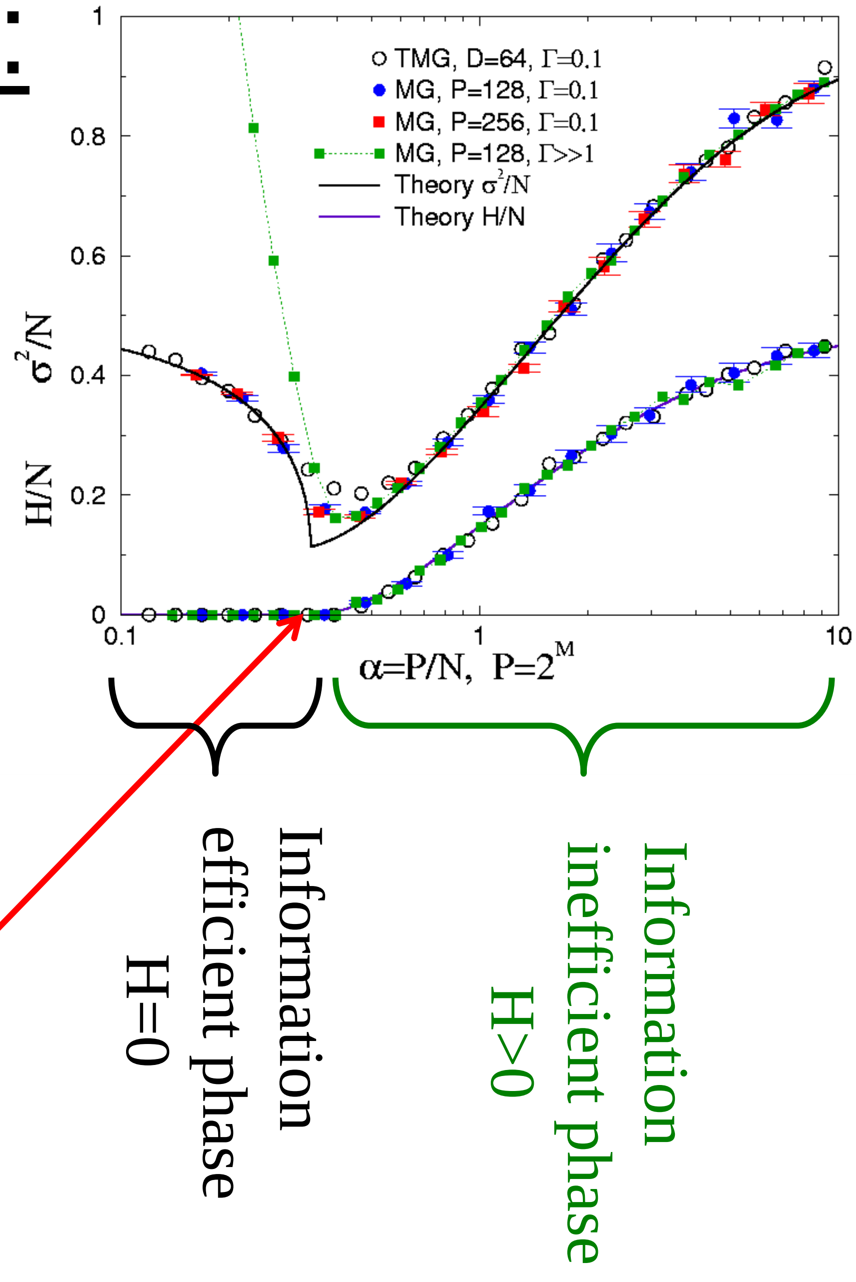
- Global efficiency

$$\sigma^2 = \langle A^2 \rangle = -\sum_{i=1}^{N} \langle u_i \rangle$$

- Predictability

$$\langle A|\mu\rangle \neq 0 \quad \Rightarrow \quad \text{predictable}$$

$$H = \frac{1}{P}\sum_{\mu=1}^{P} \langle A|\mu\rangle^2$$



Phase transition

(Challet & Marsili 1999)

Minority game with S=2

$$a_{s,i}^{\mu} = \omega_i^{\mu} + s\xi_i^{\mu}, \quad s = \pm 1 \qquad \Omega^{\mu} = \sum_{i=1}^{N} \omega_i^{\mu}$$

$$y_i(t) = \frac{\Gamma}{2}\left[U_{+1,i}(t) - U_{-1,i}(t)\right]$$

$$P\{s_i(t) = +1\} = \frac{e^{y_i(t)}}{e^{y_i(t)} + e^{-y_i(t)}}$$

Choice of agents

$$\mu(t+1) = P\mathrm{ran}()$$

Choice of Nature

$$A(t) = \Omega^{\mu(t)} + \sum_{i=1}^{N} \xi_i^{\mu(t)} s_i(t)$$

Market aggregation

$$y_i(t+1) = y_i(t) - \xi_i^{\mu(t)} \frac{A(t)}{N}$$

Learning

The stationary state is the solution of

$$\min_{\{m_i\}} H\{m_i\}$$

i.e. resources are exploited as evenly as possible by the agents

indeed $\dfrac{d\langle y_i\rangle}{dt} = -\overline{\xi_i\Omega} - \sum_{k=1}^{N}\overline{\xi_i\xi_k}m_k \quad \Rightarrow \quad \dfrac{dH}{dt} \leq 0 \quad (\langle y_i\rangle \nearrow m_i)$

replica method

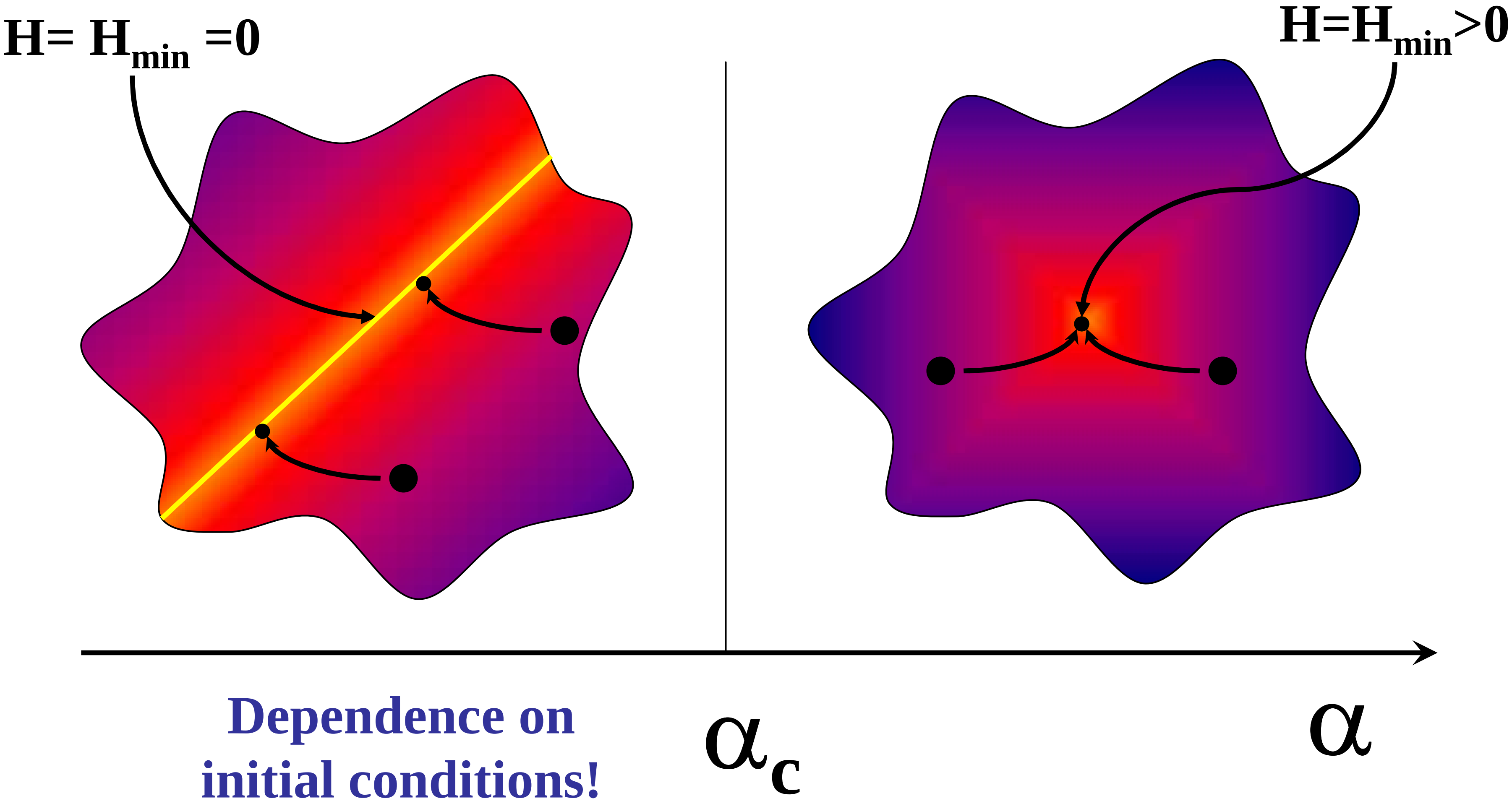
$$Z = \mathrm{Tr}_m e^{-\beta H\{m\}}$$

$$H_{\min} = \lim_{\beta\to\infty}\lim_{n\to\infty}\frac{\langle Z^n\rangle - 1}{n}$$

$\dots$ … full pdf

Phase transition

Density plot of H in the space $\{m_i\}$



Dependence on initial conditions!

Details on the calculation

- H is positive definite $\rightarrow$ replica symmetry
- Order parameters and frozen agents
- Divergence of χ, degeneracy of minima and integrated response function
- Algebraic argument for the phase transition
- How to keep replicas together …

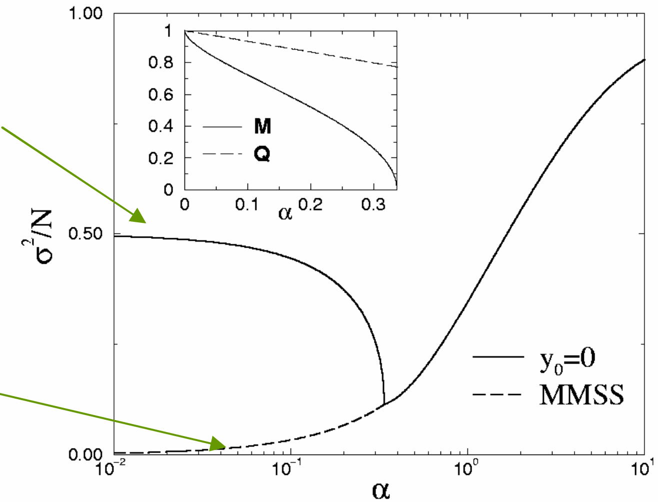
Dependence on initial conditions for $\alpha < \alpha_c$ and $\Gamma \ll 1$:

Asymmetry of initial conditions

$$y_i(0) \propto \frac{U_{+,i}(0) - U_{-,i}(0)}{2} \neq 0$$

symmetric initial conditions

Maximally asymmetric initial conditions



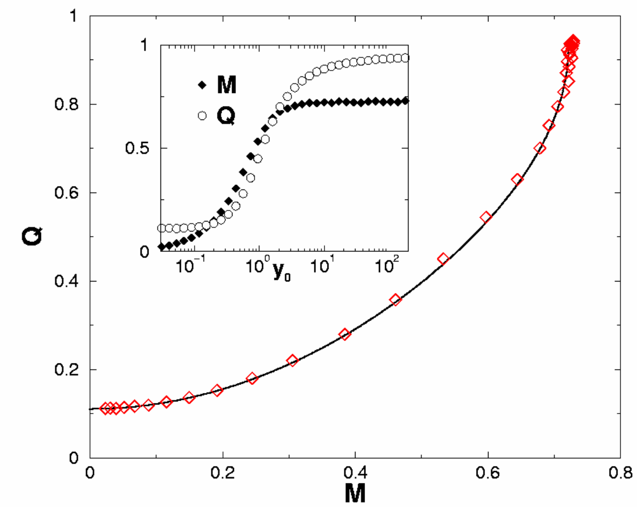
Order parameters:

$$Q = \frac{1}{N}\sum_{i=1}^{N} m_i^2,$$

$$M = \frac{1}{N}\sum_{i=1}^{N} m_i$$

$$= \frac{1}{N}\sum_{i=1}^{N} \langle s_i(t) s_i(0) \rangle$$

Overlap with initial behavior



The Nash equilibria of the Minority Game:

Replicator dynamics

$$\frac{d\pi_{i,s}}{dt} = \pi_{i,s}\left[u_i\left(s,\pi_{-i}\right) - u_i\left(\pi_i,\pi_{-i}\right)\right] = -\pi_{i,s}\sum_{j\neq i}\left[\overline{a_{i,s}\left(\pi_j\cdot a_j\right)} - \overline{\left(\pi_i\cdot a_i\right)\left(\pi_j\cdot a_j\right)}\right]$$

$$u_i\left(\pi_i,\pi_{-i}\right) = \sum_s \pi_{i,s}u_i\left(s,\pi_{-i}\right), \qquad u_i\left(s,\pi_{-i}\right) = \sum_{s_{-i}} \pi_{-i,s_{-i}}u_i\left(s,s_{-i}\right)$$

$$\sigma^2 = E\left[A^2\right] = \sum_{i,s}\pi_{i,s} + \sum_{i\neq j}\sum_{s,r}\pi_{i,s}\pi_{j,r}\overline{a_{i,s}a_{j,r}}$$

$$\frac{d\sigma^2}{dt} = \sum_{i,s}\frac{\partial\sigma^2}{\partial\pi_{i,s}}\frac{d\pi_{i,s}}{dt} = -2\sum_{i,s}\pi_{i,s}\left[\overline{\left(a_{i,s}-\pi_i\cdot a_i\right)\sum_{j\neq i}\pi_j\cdot a_j}\right]^2 \leq 0$$

σ^2 is an harmonic function of $\pi_{i,s}$ $\rightarrow$ it attains the minima on the corners
replicator dynamics converges to a pure strategy Nash equilibrium

Learning dynamics with $U_{i,s}(t+1)=U_{i,s}(t)+u_i(s,s_{-i}(t))$ also “minimizes” σ^2

What’s wrong?

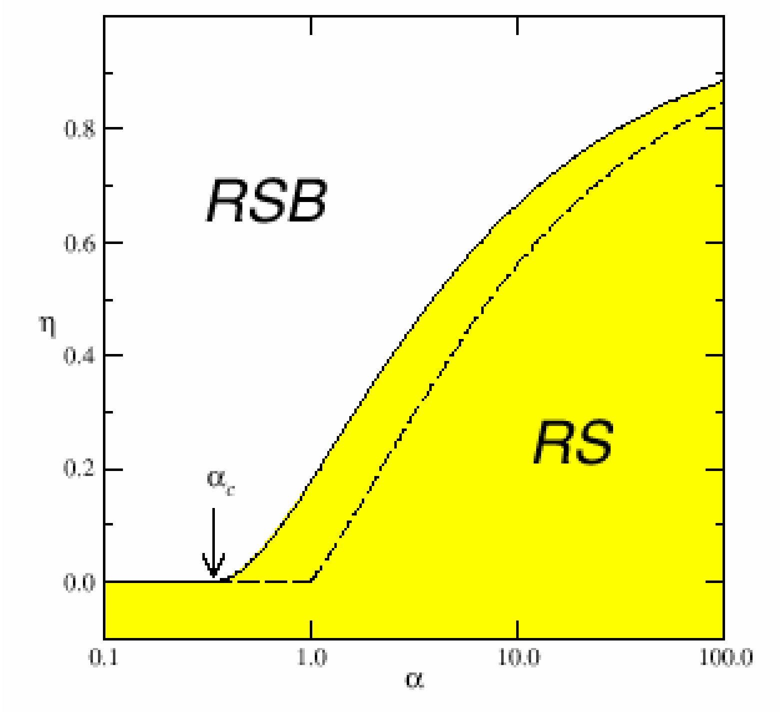
Market impact and replica symmetry breaking:

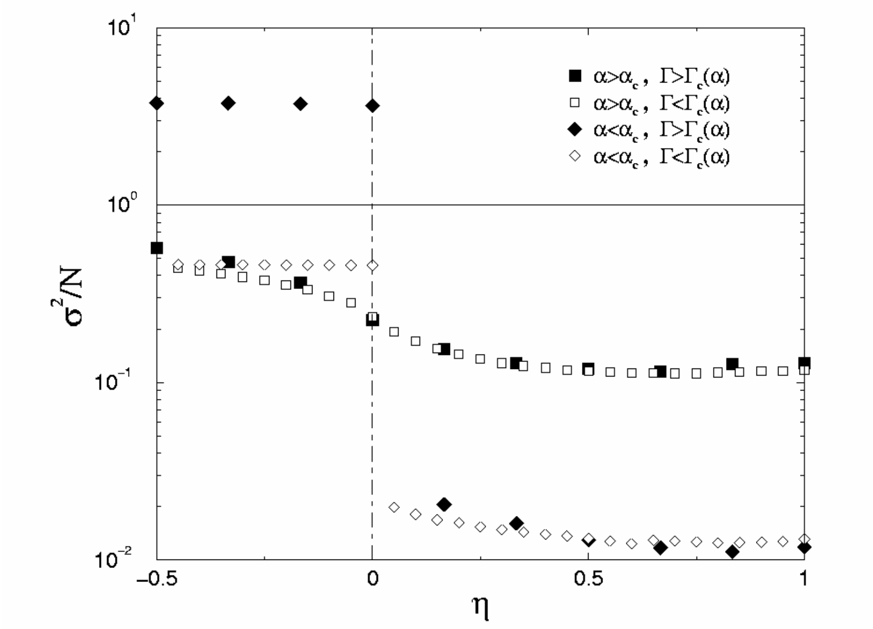
Onsager reaction term/cavity field

$$U_{s,i}(t+1) = U_{s,i}(t) - a_{s,i}^{\mu(t)}\left[A(t) - \eta\, a_{s_i(t),i}^{\mu(t)}\right]$$

⇨ Stationary state is the solution of min (1-η)H+ησ²

⇨ Replica symmetry breaking





Note:

Stylized facts occur because agents behave as price takers

If agents account for market impact

→ no anomalous fluctuations

RSB memory onset and degeneracy

- RS phase H_η has a single minimum

  $m_a=m_b \rightarrow Q = q \rightarrow \chi < \infty \rightarrow$ no memory

- RSB phase H_η has many minima

  $m_a \neq m_b \rightarrow Q > q \rightarrow \chi = \infty \rightarrow$ memory

Nash equilibria of the Minority Game

- A Nash equilibrium is a situation (in a game) where no player has incentives to change his/her behavior if others stick to theirs
- What is the best possible way to play the Minority Game?

$$g_i = \sum_{s=1}^{S} \pi_{i,s} v_{i,s}^{(0)} - 1$$

$$\Rightarrow \quad \max_{\pi} g_i \quad \Leftrightarrow \quad \pi_{i,s} = \delta_{s,s_i^*} \qquad s_i^* = \arg\max_{s} v_{i,s}^{(0)}$$

- Just play the best strategy!

Number of Nash equilibria ~ $e^{N\Sigma(\alpha)}$

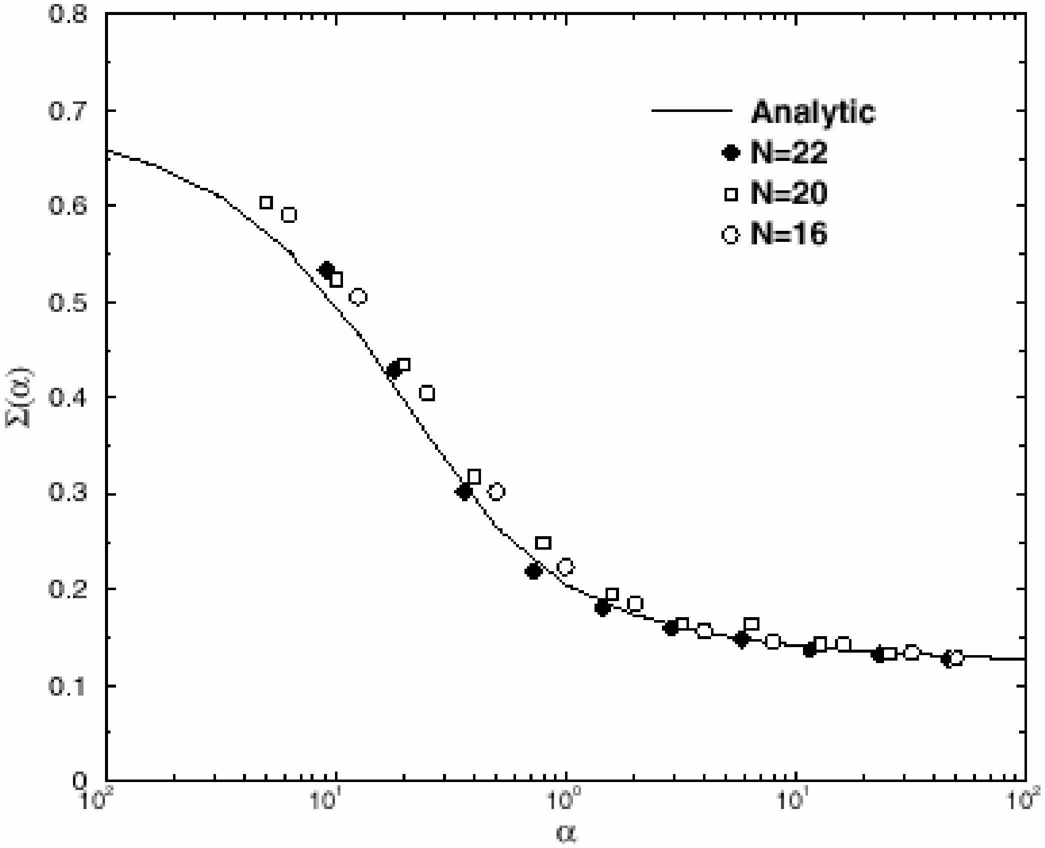


Figure 1. Logarithm of the average number of NE divided by N (Σ) as a function of α.

Generic answers:

If agents behave as

Price takers (η=0)

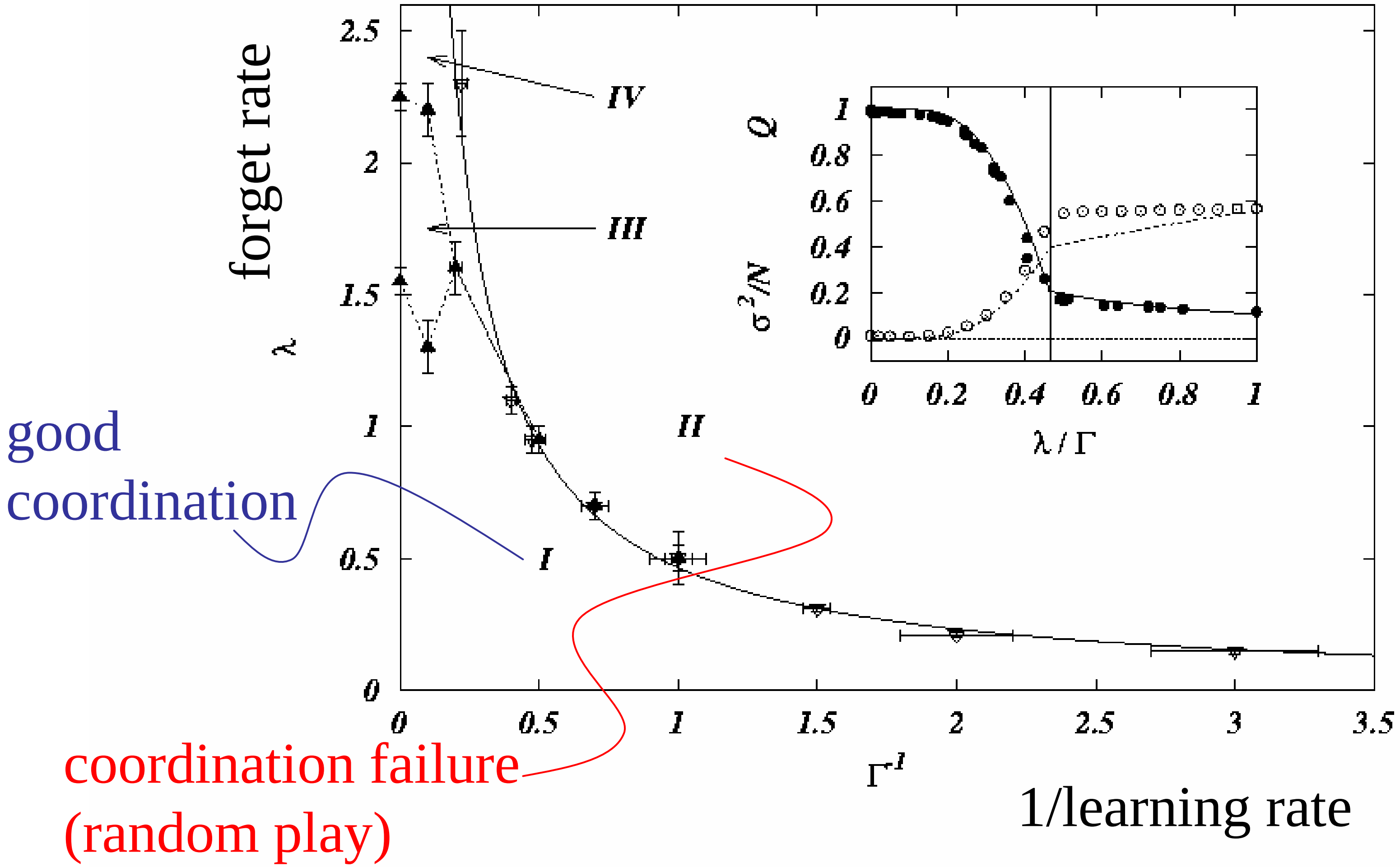
- resources are exploited - on average - as evenly as possible (min H)
- a phase transition between a symmetric (H=0) and an asymmetric (H>0) phase occurs at α_c
- Stat. state is unique if H>0 or a continuum for H=0
- Agents behavior is stochastic (σ^2>H). They can be worse off if they have more choices
- ...

Strategic players (η=1)

- Fluctuations are as small as possible (min σ^2)
- No phase transition occurs (H>0 for all α>0)
- Many ($e^{\phi N}$) disconnected states exist
- Agents behavior is deterministic (σ^2=H). They are always better off if they are given more alternatives
- ...

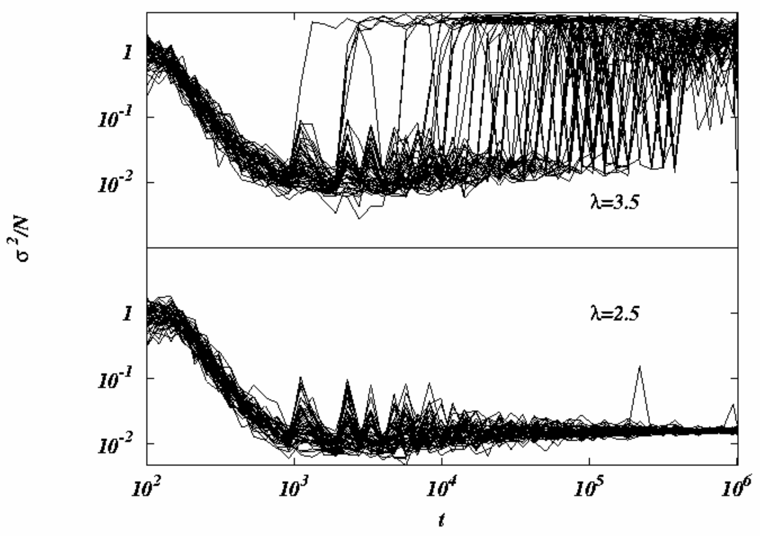
Learning to coordinate with finite memory

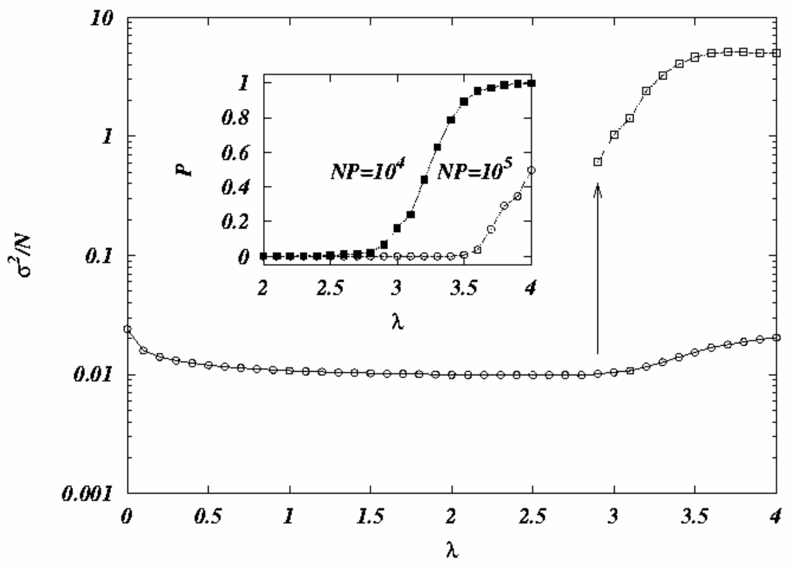
$$U_{s,i}(t+1)=(1-\lambda)U_{s,i}(t)-a^{\mu}_{s,i}\left[A(t)-a^{\mu}_{s_i(t),i}\right]$$



But in a changing world:

Behavioral rules are re-drawn at a small rate





→ transition becomes discontinuous
small changes can have catastrophic consequences!