A New Model for Selfish Routing*

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Abstract

In this work, we introduce and study a new, potentially rich model for *selfish routing* over *non-cooperative networks* as an interesting hybridization of the two best studied such models; these are the more recent KP *model* due to Koutsoupias and Papadimitriou [20] and the much older W *model* due to Wardrop [30].

In our new model, we consider a set of n *users,* each using a *mixed strategy* to ship its unsplittable *traffic* over a *network* consisting of m parallel *links*. In a *Nash equilibrium,* no *user* can decrease its *Expected Individual Cost* by unilaterally deviating from its strategy. To evaluate the performance of Nash equilibria, we introduce *Quadratic Social Cost* as the *sum* of the expectations of the *latencies* incurred by the squares of the accumulated traffics. This definition is unlike the KP model, where *Maximum Social Cost* [20] has been defined as the expectation of the *maximum* latency incurred by the accumulated traffics. To the contrary, this definition is aligned with the W model, as it happens that Quadratic Social Cost is a weighted sum of Expected Individual Costs. We use Quadratic Social Cost to define and study *Quadratic Coordination Ratio,* much in the same way *Maximum Coordination Ratio* [20] has been defined using Maximum Social Cost.

We are interested in understanding the impact of our modeling assumptions on the various algorithmic, combinatorial, structural and optimality properties of Nash equilibria in this new model. The main findings of our study are outlined as follows:

- We prove that Quadratic Social Cost can be computed in polynomial time. This stands in contrast to the $\#\mathcal{P}$-completeness of computing Maximum Social Cost [14].
- Call *worst-case* Nash equilibrium the one that maximizes Quadratic Social Cost. Through an elegant, combinatorial analysis, we prove that for the case of identical users and identical links, the *worst-case* Nash equilibrium is the *fully mixed Nash equilibrium* [23], where each user assigns (strictly) positive probability to every link.
- As our main result, we present a comprehensive collection of *tight,* constant (that is, independent of m and n) lower and upper bounds on Quadratic Coordination Ratio for a number of interesting special cases of our model. All the bounds we prove are strictly less than 2; moreover, some of them stand in contrast to corresponding $\Theta\left(\frac{\lg m}{\lg \lg m}\right)$ bounds on Maximum Coordination Ratio, which were previously shown in [9, 19, 20, 23], for the KP model.

1 Introduction

1.1 Motivation and Framework

1.1.1 Outline

In this work, we propose a new model for *selfish routing* over *non-cooperative networks*. This model is an interesting hybridization of the two most famous models for selfish routing that were studied before; these are the so called KP *model* due to Koutsoupias and Papadimitriou [20] and the W *model* due to Wardrop [30]. The KP model was proposed only recently in the context of studying the effects of selfish traffic over the Internet; however, it has yet received a lot of flourishing interest and attention – see, e.g., [9, 8, 11, 12, 14, 15, 18, 19, 22, 23] or [13] for a survey. In contrast, the W model dates back to the 1950s, when it was used for studying the economics of transportation networks (see, e.g., [3, 4, 10]). However, much recent work on selfish routing (see [29] and references therein) has adopted the W model and witnessed a revival of interest into it.

Within our new model, we study some interesting *strategic game*, which was originally proposed in [20]; we are especially interested in the associated *Nash equilibria* [25, 26]. At a Nash equilibrium, no player (here, *user*) can unilaterally improve its objective by switching to a different *strategy* (here, *link*); arguably, Nash equilibrium is the most robust solution concept in (*Non-Cooperative*) *Game Theory* [27]. In a *pure* Nash equilibrium, each player chooses exactly one strategy (with probability one); in a *mixed* Nash equilibrium, each player uses a probability distribution over strategies, and in the *fully mixed* Nash equilibrium [23] all probabilities are (strictly) positive. Nash equilibria enjoy some remarkable properties; most notably, there is always a mixed Nash equilibrium [26] for any finite strategic game.

We embark on a systematic study of some interesting algorithmic, combinatorial, structural and optimality properties of Nash equilibria for the routing game we consider within our proposed hybrid model for selfish routing.

1.1.2 The KP Model and the W Model

The two models differ with respect to their assumptions about the following parameters:

- The structure and topology of the underlying *network* they consider; however, in both cases, two distinguished nodes, the *source* and the *destination*, are considered.

- The *splittability* or *unsplittability* of the selfish traffic; *unsplittable* traffic is routed all together along a single path, while *splittable* traffic may split into infinitesimal pieces.

- The type of equilibria (*pure* or *mixed*) they consider.

- The definition of *(Expected) Individual Cost* they adopt; these are used for defining Nash equilibria and *Wardrop equilibria,* respectively.

- The definitions of *Social Cost* and *Optimum* – a performance measure for (Nash or Wardrop) equilibria, and an optimality measure for assignments (not necessarily equilibria), respectively.

  In either model, these two definitions give rise to *Coordination Ratio,* also known as *Price of Anarchy* [28]; this is the *worst-case* ratio of Social Cost over Optimum. A *worst-case* Nash equilibrium is one that maximizes the particular Social Cost.

We continue to describe separately the KP model and the W model.

- The KP model has considered a simple network consisting of *m parallel links* from *source* to *destination*; each link bears a *capacity* – the rate at which the link processes traffic. Selfish traffic is modeled as a finite collection of *users.* Each user bears an unsplittable traffic, which it ships using a *mixed strategy* – a probability distribution over links; thus, mixed Nash equilibria have been accomodated in the KP model. The *Expected Individual Cost* of a user is its expected latency; in a Nash equilibrium, Expected Individual Cost cannot be unilaterally decreased. The Social Cost in a Nash equilibrium, called *Maximum Social Cost,* is the expectation, over all random choices of the users, of the maximum, over all links, latency on a link; the Optimum, also called the *Maximum Optimum,* is the least possible maximum, over all links, latency on a link. The Coordination Ratio is called *Maximum Coordination Ratio.* Alternatively, the KP model may be viewed as a *weighted congestion game* [24] equipped with Social Cost and Coordination Ratio.

- The W model has considered arbitrary networks with a *source* and a *destination,* and a *latency function* for each link. (The parallel links network from the KP model is a special case.) Selfish traffic is modeled as a *splittable flow.* The *Individual Cost* for a path (from source to destination) is the sum of the latencies incurred on its links. In a *Wardrop equilibrium,* all (used) paths have the same Individual Cost. This is like a Nash equilibrium in the KP model, but with infinitely many users, each carrying infinitesimal traffic. For this reason, the W model has been restricted to pure equilibria. The Social Cost in a Wardrop equilibrium is the sum, over all paths, of Individual Costs. The Optimum is the least possible, over all flows, Social Cost.

1.1.3 The New Model

In the new model, we adopt the parallel links network considered in the KP model. Moreover, we follow the KP model to consider unsplittable traffic and mixed Nash equilibria. The Expected

Individual Cost we adopt is the one used in the KP model – the expected latency incurred to a user. Hence, the Nash equilibria for our new model are exactly those for the KP model. However, we follow the W model to model Social Cost in such a way that it turns out to be a certain sum of Expected Individual Costs. So, in a sense, our model is the W model restricted to the parallel links network but modified to host unsplittable traffic and mixed strategies; these two features are borrowed from the KP model. We note that our new model represents the *first* step towards accommodating unsplittable traffic within the W model.

We continue to describe our model in some more detail. For any link, consider the latency incurred by the square of the total traffic on the link; take the expectation of this latency and add up over all links. This results to the Social Cost for our model, which we call *Quadratic Social Cost*. (Similar modelings that have employed quadratic cost functions can be found in both the scheduling literature [1, 6, 7, 21] and the networking literature [2].) Naturally, *Quadratic Optimum* is the best possible Quadratic Social Cost, over all pure assignments. *Quadratic Coordination Ratio* is the worst-case ratio, over all Nash equilibria, of Quadratic Social Cost over Quadratic Optimum.

Note that expectation and maximum used in the definition of Maximum Social Cost for the KP model do not commute. However, expectation and sum used in the definition of Quadratic Social Cost for our model do commute. This available commutativity allows some hope for a more tractable analysis for several algorithmic, combinatorial, structural and optimality properties of Nash equilibria in our new model.

1.2 Contribution and Significance

Some of our results deal specifically with the cases of *identical users* and *identical links* with equal (unit) user traffics and link capacities, respectively; the most general cases are those of *arbitrary users* and *related links,* respectively. We partition our results into two major groups around Quadratic Social Cost and Quadratic Coordination Ratio, respectively.

1.2.1 Quadratic Social Cost

We observe that Quadratic Social Cost can be expressed as a weighted sum of Expected Individual Costs, where the weights are the user traffics. This observation allows for the computation of Quadratic Social Cost for any mixed assignment in polynomial time $O(mn)$ (Theorem 4.1).

For the special case of fully mixed Nash equilibria, our general analysis yields two very simple combinatorial expressions for Quadratic Social Cost in the two separate cases of identical links and identical users (Theorems 4.2 and 4.4, respectively). For these two cases, the two expressions immediately imply corresponding polynomial time algorithms for the computation of Quadratic Social Cost (Corollaries 4.3 and 4.5).

For the case of identical users, the link capacities enter the combinatorial expression for Quadratic Social Cost via their sum; moreover, Quadratic Social Cost is inversely proportional to the sum. This form of dependence is significant since it excludes any possible *Braess-like paradoxes* [5], where increasing a link capacity results to an increase of the Quadratic Social Cost for the fully mixed Nash equilibrium. In addition, this form of dependence implies that the Quadratic Social Cost of the fully mixed Nash equilibrium is insensitive to reallocating capacity among the links.

A natural question is to identify the worst-case Nash equilibrium with respect to Quadratic Social Cost. We address this question for the special case of identical users and identical links. We use an elegant, combinatorial analysis to prove that the worst-case Nash equilibrium is the fully mixed Nash equilibrium (Theorem 4.8). We formulate the *Quadratic Fully Mixed Nash Equilibrium Conjecture* (Conjecture 4.1) to speculate that this happens in the general case.

1.2.2 Quadratic Coordination Ratio

We proceed to study the Quadratic Coordination Ratio, for which we obtain, as our main result, a collection of *tight* bounds. All bounds we prove are either constant or bounded by a constant – independent of m and n. All constants are strictly less than 2.

- We first consider pure Nash equilibria. (Pure Nash equilibria always exist in our model since such are known to exist for the KP model [14, Theorem 1] and the two coincide.) To establish our upper bounds on Quadratic Coordination Ratio for them, we prove some new structural properties of optimal assignments and pure Nash equilibria for the specific settings we consider, which may be of independent interest. For example, Proposition 3.2 provides an intuitive characterization of optimal assignments in the new model as assignments that cannot be improved locally by switching one user from one link to another. This characterization immmediately implies that optimal assignments in the new model can be decided in polynomial time (Corollary 3.3). We use such new properties of optimal assignments and pure Nash equilibria to formalize the maximization of Quadratic Coordination Ratio as a non-linear maximization problem, in turn, we solve it to obtain an upper bound on Quadratic Coordination Ratio. Our specific findings are as follows.
  - For the case of identical users, the Quadratic Coordination Ratio is $\frac{4}{3}$ (Theorem 5.1).
  - For the case of identical links, the Quadratic Coordination Ratio is $\frac{9}{8}$ (Theorem 5.5).

  The different tight bounds of $\frac{4}{3}$ and $\frac{9}{8}$ imply a Quadratic Coordination Ratio separation between the cases of identical links and of identical users, respectively, restricted to pure

Nash equilibria.

We believe that the technique of expressing the maximization of Quadratic Coordination Ratio as a non-linear maximization problem, and subsequently solving it, may be of further interest and applicability.

- We continue to consider mixed Nash equilibria.
  - For the case of identical links, we prove a *tight* bound of $2 - \max\left\{\frac{1}{n}, \frac{1}{m}\right\}$ on Quadratic Social Cost for the fully mixed Nash equilibrium.
  - For the case of identical users and identical links, we prove a sometimes better upper bound of $1 + \min\left\{\frac{m-1}{n}, \frac{n-1}{m}\right\}$, which, however, holds for all Nash equilibria.

1.3 Directly Related Work and Comparison

Computing the Maximum Social Cost of an arbitrary Nash equilibrium in the KP model is known to be $\#\mathcal{P}$-complete [14, Theorem 8], and this applies even if links are identical. This stands in contrast to the obtained polynomial computation of Quadratic Social Cost in the new model.

The works by Fotakis *et al.* [14], by Gairing *et al.* [18], and by Lücking *et al.* [22] dwelved into the combinatorial structure and the computational complexity of Nash equilibria for the KP model. Fully mixed Nash equilibria were introduced and analyzed in [23]. The (yet unproven) *Fully Mixed Nash Equilibrium Conjecture* asserts that the worst-case Nash equilibrium for the KP model is the fully mixed Nash equilibrium. This conjecture has been motivated by some results in [14]; it was explicitly formulated in [18] and further studied in [16, 17, 22].

Bounds on Maximum Coordination Ratio for the KP model were proved in [9, 12, 15, 19, 23]. These include (*tight*) bounds of $\Theta\left(\frac{\lg m}{\lg\lg m}\right)$ for the case of identical links [9, 19, 20, 23] and of $\Theta\left(\frac{\lg m}{\lg\lg\lg m}\right)$ for the case of related links [9]. Some of these bounds stand in contrast to the corresponding constant bounds on Quadratic Coordination Ratio proved in this work for the new model. However, for the W model, there have been shown constant bounds on Coordination Ratio – see [29] and references therein.

1.4 Organization

The rest of this paper is organized as follows. Section 2 presents our definitions and some preliminaries. Some preliminary properties of Nash equilibria and optimal (pure) assignments are articulated in Section 3. Quadratic Social Cost is studied in Section 4. The bounds on Quadratic Coordination Ratio are presented in Section 5. We conclude, in Section 6, with a discussion of our results and some open problems.

2 The New Model

Our definitions for the new model are built on top of those for the KP model [20]; those definitions are extended to accomodate features from the W model [30]. The definitions for the KP model are patterned after those in [23, Section 2], [14, Section 2], [12, Section 2], [18, Section 2] and [22, Section 2], which, in turn, were based on those in [20, Sections 1 & 2].

Throughout, denote for any integer $m \geq 1$, $[m] = \{1, \ldots, m\}$; take $[0] = \emptyset$. For a random variable X with associated probability distribution $\mathbf{P}$; denote $\mathrm{E}_{\mathbf{P}}(X)$ the *expectation* of X.

2.1 General

We consider a simple *network* consisting of a set of m parallel *links* $1, 2, \ldots, m$ from a *source* node to a *destination* node. Each of n *users* $1, 2, \ldots, n$ wishes to route a particular amount of traffic along a (non-fixed) link from source to destination. Assume throughout that $m \geq 2$ and $n \geq 2$. (Throughout, we will be using subscripts for users and superscripts for links.)

Denote w_i the *traffic* of user $i \in [n]$. Define the $n \times 1$ *traffic vector* $\mathbf{w}$ in the natural way. Without loss of generality, assume that $w_1 \geq w_2 \geq \ldots \geq w_n$. Denote $W = \sum_{i \in [n]} w_i$, $W_1 = \sum_{i \in [n]} w_i^2$ and $W_2 = \sum_{i,j \in [n], i<j} w_i w_j$. (These quantities will be used in our later proofs.) Note that $W^2 = W_1 + 2W_2$. It is a well known simple fact that $W_1 \geq \frac{W^2}{n}$.

Denote $c^\ell > 0$ the *capacity* of link $\ell \in [m]$, representing the rate at which the link processes traffic. So, the *latency* for traffic w through link ℓ equals w/c^ℓ. Assume throughout, without loss of generality, that $c^1 \geq c^2 \geq \ldots \geq c^m$. Denote $C = \sum_{j \in [m]} c^j$. An *instance* is a pair $\langle \mathbf{w}, \mathbf{c} \rangle$.

In the case of *identical users,* all user traffics are 1; in the case of *identical links,* all link capacities are 1. In the general case, we talk about *arbitrary users* and *related links.*

2.2 Strategies and Assignments

A *pure strategy* for user $i \in [n]$ is some specific link. A *mixed strategy* for user $i \in [n]$ is a probability distribution over pure strategies; so, it is a probability distribution over links.

A *pure assignment,* is an n-tuple $\mathbf{L} = \langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n$; a *mixed assignment,* is an $n \times m$ *probability matrix* $\mathbf{P}$ of nm probabilities p_i^j, $i \in [n]$ and $j \in [m]$, where p_i^j is the probability that user i chooses link j. Throughout, we will cast a pure assignment as a special case of a mixed assignment in which all (mixed) strategies are pure. Moreover, the mixed assignment $\mathbf{P}$ can be cast as a collection of pure assignments $\mathbf{L}_{\mathbf{P}} = \langle \ell_1, \ell_2, \ldots, \ell_n \rangle$ such that for all users $i \in [n]$, $p_i^{\ell_i} > 0$; any such pure assignment $\mathbf{L}_{\mathbf{P}}$ will be called *consistent* with $\mathbf{P}$. So, a mixed assignment $\mathbf{P}$ induces a probability distribution $\mathsf{Pr}_{\mathbf{P}}$ on the set of all pure assignments that are consistent with $\mathbf{P}$. A mixed assignment $\mathbf{P}$ is *fully mixed* [23, Section 2.2] if for all users $i \in [n]$ and links

$j \in [m]$, $p_i^j > 0$. A user $i \in [n]$ is *solo* in the pure assignment $\mathbf{L}$ if no other user user is assigned to link ℓ_i; in such case, say that link ℓ_i is *solo* as well.

Fix now a mixed assignment $\mathbf{P}$. The *load* $\delta^\ell(\mathbf{P})$ on link $\ell \in [m]$ induced by $\mathbf{P}$ is the total traffic assigned to the link according to $\mathbf{P}$; so, $\delta^\ell(\mathbf{P})$ is a random variable. For each link $\ell \in [m]$, denote $\theta^\ell(\mathbf{P})$ the *expected load* on link $\ell \in [m]$; thus, $\theta^\ell(\mathbf{P}) = \mathrm{E}_{\mathbf{P}}(\delta^\ell(\mathbf{P})) = \sum_{i\in[n]} p_i^\ell w_i$. The *latency* $\Delta^\ell(\mathbf{P})$ on link $\ell \in [m]$ induced by $\mathbf{P}$ is the latency due to the load assigned to the link according to $\mathbf{P}$; so, $\Delta^\ell(\mathbf{P})$ is a random variable and $\Delta^\ell(\mathbf{P}) = \frac{\delta^\ell(\mathbf{P})}{c^\ell}$. For each link $\ell \in [m]$, denote $\Theta^\ell(\mathbf{P})$ the *expected latency* on link $\ell \in [m]$; thus, $\Theta^\ell(\mathbf{P}) = \frac{\theta^\ell(\mathbf{P})}{c^\ell}$.

2.3 Costs

2.3.1 Individual Cost and Expected Individual Cost

For a pure assignment $\mathbf{L}$, the *Individual Cost* for user $i \in [n]$, denoted $\lambda_i(\mathbf{L})$, is $\lambda_i(\mathbf{L}) = \Delta^{\ell_i}(\mathbf{L})$; so, the Individual Cost for a user is the latency of the link it chooses. For a mixed assignment $\mathbf{P}$, the *Expected Individual Cost* for user $i \in [n]$, denoted $\lambda_i(\mathbf{P})$, is the expectation according to $\mathbf{P}$ of the Individual Cost for the user in any pure assignment $\mathbf{L_P}$; so, $\lambda_i(\mathbf{P}) = \mathrm{E}_{\mathbf{P}}(\lambda_i(\mathbf{L_P}))$.

2.3.2 Quadratic Social Cost

Associated with an instance $\langle \mathbf{w}, \mathbf{c}\rangle$ and a mixed assignment $\mathbf{P}$ is the *Quadratic Social Cost,* denoted as $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$, which is defined as the expectation according to $\mathbf{P}$ of the sum of the link latencies due to the squares of the incurred loads in a pure assignment $\mathbf{L}$ consistent with $\mathbf{P}$; so,

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \quad = \quad \mathrm{E}_{\mathbf{P}}\left(\sum_{\ell\in[m]} \frac{\left(\delta^\ell(\mathbf{L_P})\right)^2}{c^\ell}\right).$$

For a pure assignment $\mathbf{L_P} = \langle \ell_1, \ldots, \ell_n\rangle$, changing the summation order and using the definition of Individual Cost yields that $\sum_{\ell\in[m]} \frac{\left(\delta^\ell(\mathbf{L_P})\right)^2}{c^\ell} = \sum_{i\in[n]} w_i \cdot \frac{\delta^{\ell_i}(\mathbf{L_P})}{c^{\ell_i}} = \sum_{i\in[n]} w_i \lambda_i(\mathbf{L_P})$. So,

$$\begin{aligned}
&\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \\
= \;& \mathrm{E}_{\mathbf{P}}\left(\textstyle\sum_{i\in[n]} w_i \lambda_i(\mathbf{L_P})\right) \\
= \;& \textstyle\sum_{i\in[n]} w_i \, \mathrm{E}_{\mathbf{P}}(\lambda_i(\mathbf{L_P})) \quad \text{(by linearity of expectation)} \\
= \;& \textstyle\sum_{i\in[n]} w_i \, \lambda_i(\mathbf{P}) \qquad \text{(by definition of Expected Individual Cost)}
\end{aligned}$$

Thus, the Quadratic Social Cost is a weighted sum of Expected Individual Costs, where the weights are the user traffics. There is a counterpart of Quadratic Social Cost in the KP model. Associated with an instance $\langle \mathbf{w}, \mathbf{c}\rangle$ and a mixed assignment $\mathbf{P}$ is the *Maximum Social Cost* [20], denoted $\mathsf{MSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$, which is defined as the expectation according to $\mathbf{P}$ of the maximum of the incurred link latencies; so, $\mathsf{MSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = \mathrm{E}_{\mathbf{P}}\left(\max_{\ell \in [m]} \frac{\delta^\ell(\mathbf{L})_{\mathbf{P}}}{c^\ell}\right)$.

2.3.3 Quadratic Optimum

Associated with an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ is the *Quadratic Optimum,* denoted $\mathsf{QOPT}(\mathbf{w}, \mathbf{c})$, which is defined as the least possible, over all pure assignments, sum of link latencies incurred by the squares of the total traffics on the links; so,

$$\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) \quad = \quad \min_{\mathbf{L} \in [m]^n} \sum_{\ell \in [m]} \frac{\left(\delta^\ell(\mathbf{L})\right)^2}{c^\ell}.$$

Note that $\mathsf{QOPT}(\mathbf{w}, \mathbf{c})$ refers to the *optimal* pure assignment. Formally, a pure assignment $\mathbf{L}$ is *optimal* for the instance $\langle \mathbf{w}, \mathbf{c} \rangle$ if $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) = \mathsf{QOPT}(\mathbf{w}, \mathbf{c})$. Clearly, for the case of identical users and identical links, $\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) \geq \max\left\{n, \frac{n^2}{m}\right\} = n \max\left\{1, \frac{n}{m}\right\}$. There is a counterpart of Quadratic Optimum in the KP model. This is the *Maximum Optimum* [20], denoted $\mathsf{MOPT}(\mathbf{w}, \mathbf{c})$, defined as the least possible, over all pure assignments, maximum incurred link latency; so, $\mathsf{MOPT}(\mathbf{w}, \mathbf{c}) = \min_{\mathbf{L} \in [m]^n} \max_{\ell \in [m]} \frac{\delta^\ell(\mathbf{L})}{c^\ell}$.

2.4 Nash Equilibria

We are interested in a special class of (mixed) assignments called Nash equilibria [26] that we describe below. Formally, given an instance $\langle \mathbf{w}, \mathbf{c} \rangle$, the mixed assignment $\mathbf{P}$ is a *Nash equilibrium* [20, Section 2] if for each user $i \in [n]$, it minimizes the Expected Individual Cost $\lambda_i(\mathbf{P})$, over all mixed assignments $\mathbf{Q}$ that differ with $\mathbf{P}$ only with respect to the mixed strategy of user i; that is, for all such mixed assignments $\mathbf{Q}$, $\lambda_i(\mathbf{P}) \leq \lambda_i(\mathbf{Q})$. Thus, in a Nash equilibrium, there is no incentive for a user to unilaterally deviate from its mixed strategy.

Denote a fully mixed Nash equilibrium as $\mathbf{F}$. It is known [23, Lemma 15] that in the case of identical links, a fully mixed Nash equilibrium exists always and uniquely; there, all links are equiprobable (each chosen with probability $\frac{1}{m}$) for each user. It is also known [23, Lemma 22] that the fully mixed Nash equilibrium does not always exist in the case of identical users; however, when it exists, it is unique but links are no more equiprobable – for each user, each link $\ell \in [m]$ is chosen with probability $\frac{m+n-1}{(n-1)C}\, c^\ell - \frac{1}{n-1}$.

2.5 Quadratic Coordination Ratio

The *Quadratic Coordination Ratio,* denoted QCR, is the worst-case ratio $\dfrac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$, over all instances $\langle \mathbf{w}, \mathbf{c} \rangle$ and associated Nash equilibria $\mathbf{P}$. This is similar to *Maximum Coordination Ratio,* denoted MCR, which was defined in [20] as the worst-case ratio $\dfrac{\mathsf{MSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{MOPT}(\mathbf{w}, \mathbf{c})}$, over all instances $\langle \mathbf{w}, \mathbf{c} \rangle$ and associated Nash equilibria $\mathbf{P}$,

3 Preliminaries

We present some properties of Nash equilibria and optimal assignments in the new model. We first prove that Expected Individual Cost takes a special form for any (mixed) assignment.

Lemma 3.1 *Consider an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and an associated mixed assignment $\mathbf{P}$. Then, for each user $i \in [n]$,*

$$\lambda_i(\mathbf{P}) \;=\; \sum_{\ell \in [m]} p_i^{\ell} \frac{w_i + \sum_{k \in [n]: k \neq i} p_k^{\ell} w_k}{c^{\ell}} .$$

Proof: By the definition of Individual Cost and Expected Individual Cost, we obtain that

$$\begin{aligned}
\lambda_i(\mathbf{P}) &= \mathrm{E}_{\mathbf{P}}(\lambda_i(\mathbf{L_P})) \\
&= \sum_{\mathbf{L_P} \in [m]^{[n]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \lambda_i(\mathbf{L_P}) \\
&= \sum_{j \in [m]} \left(\sum_{\mathbf{L_P} \in [m]^{[n]} \,:\, l_i = j} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \lambda_i(\mathbf{L_P}) \right) \\
&= \sum_{j \in [m]} \left(\sum_{\mathbf{L_P} \in [m]^{[n] \setminus \{i\}} \,:\, l_i = j} p_i^j \, \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \lambda_i(\mathbf{L_P}) \right) \\
&= \sum_{j \in [m]} p_i^j \left(\sum_{\mathbf{L_P} \in [m]^{[n] \setminus \{i\}} \,:\, l_i = j} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \lambda_i(\mathbf{L_P}) \right) \\
&= \sum_{j \in [m]} p_i^j \left(\sum_{\mathbf{L_P} \in [m]^{[n] \setminus \{i\}}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \lambda_i(\mathbf{L_P}) \mid_{l_i = j} \right) \\
&= \sum_{j \in [m]} p_i^j \left(\sum_{\mathbf{L_P} \in [m]^{[n] \setminus \{i\}}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \left(\frac{\sum_{k \in [n] : l_k = j} w_k}{c^j} \right) \right) \\
&= \sum_{j \in [m]} p_i^j \left(\sum_{\mathbf{L_P} \in [m]^{[n] \setminus \{i\}}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \left(\frac{w_i + \sum_{k \in [n] \setminus \{i\} : l_k = j} w_k}{c^j} \right) \right) \\
&= \sum_{j \in [m]} p_i^j \left(\frac{w_i}{c^j} \sum_{\mathbf{L_P} \in [m]^{[n] \setminus \{i\}}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) + \sum_{\mathbf{L_P} \in [m]^{[n] \setminus \{i\}}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \left(\frac{\sum_{k \in [n] \setminus \{i\} : l_k = j} w_k}{c^j} \right) \right) \\
&= \sum_{j \in [m]} p_i^j \left(\frac{w_i}{c^j} + \sum_{\mathbf{L_P} \in [m]^{[n] \setminus \{i\}}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \left(\frac{\sum_{k \in [n] \setminus \{i\} : l_k = j} w_k}{c^j} \right) \right)
\end{aligned}$$

Reorder users so that $i = n$. We prove by backward induction on r, $0 \leq r \leq n-1$, that

$$\lambda_i(\mathbf{P}) \;=\; \sum_{j \in [m]} p_i^j \left(\frac{w_i}{c^j} + \sum_{k \in [n-1] \setminus [r]} \frac{p_k^j w_k}{c^j} + \sum_{\mathbf{L_P} = \langle l_1, \ldots, l_r \rangle \in [m]^{[r]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \left(\frac{\sum_{k \in [r] : \ell_k = j} w_k}{c^j} \right) \right) .$$

For the basis case, where $r = n-1$, $\sum_{k \in [n-1] \setminus [r]} \frac{p_k^j w_k}{c^j} = 0$; so, the claim reduces to

$$\lambda_i(\mathbf{P}) \;=\; \sum_{j \in [m]} p_i^j \left(\frac{w_i}{c^j} + \sum_{\mathbf{L_P} \in [m]^{[n-1]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \cdot \left(\frac{\sum_{k \in [n-1] : \ell_k = j} w_k}{c^j} \right) \right) ,$$

which has been shown. Assume the claim for r, $0 < r \leq n-1$. and prove it for $r-1$. So,

$$\begin{aligned}
& \sum_{\mathbf{L_P} \in [m]^{[r]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k \in [r] : \ell_k = j} w_k}{c^j} \right) \\
= & \sum_{\substack{\mathbf{L_P} \in [m]^{[r]} : \\ l_r = j}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k \in [r] : \ell_k = j} w_k}{c^j} \right) |_{l_r = j} \\
& + \sum_{\substack{\mathbf{L_P} \in [m]^{[r]} : \\ l_r \neq j}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k \in [r] : \ell_k = j} w_k}{c^j} \right) |_{l_r \neq j} \\
= & \sum_{\substack{\mathbf{L_P} \in [m]^{[r]} : \\ l_r = j}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{w_r + \sum_{k \in [r-1] : \ell_k = j} w_k}{c^j} \right) \\
& + \sum_{\substack{\mathbf{L_P} \in [m]^{[r]} : \\ l_r \neq j}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k \in [r-1] : \ell_k = j} w_k}{c^j} \right) \\
= & \sum_{\mathbf{L_P} \in [m]^{[r-1]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P})\, p_r^j \left(\frac{w_r + \sum_{k \in [r-1] : \ell_k = j} w_k}{c^j} \right) \\
& + \sum_{\mathbf{L_P} \in [m]^{[r-1]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P})\, (1 - p_r^j) \left(\frac{\sum_{k \in [r-1] : \ell_k = j} w_k}{c^j} \right) \\
= & \sum_{\mathbf{L_P} \in [m]^{[r-1]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{p_r^j w_r}{c^j} + \frac{\sum_{k \in [r-1] : \ell_k = j} w_k}{c^j} \right) \\
= & \frac{p_r^j w_r}{c^j} \sum_{\mathbf{L_P} \in [m]^{[r-1]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) + \sum_{\mathbf{L_P} \in [m]^{[r-1]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k \in [r-1] : \ell_k = j} w_k}{c^j} \right) \\
= & \frac{p_r^j w_r}{c^j} + \sum_{\mathbf{L_P} \in [m]^{[r-1]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k \in [r-1] : \ell_k = j} w_k}{c^j} \right) .
\end{aligned}$$

Hence, by induction hypothesis, it follows that

$$\begin{aligned}
&\lambda_i(\mathbf{P}) \\
&= \sum_{j\in[m]} p_i^j \left(\frac{w_i}{c^j} + \sum_{k\in[n-1]\setminus[r]} \frac{p_k^j w_k}{c^j} + \frac{p_r^j w_r}{c^j} + \sum_{\mathbf{L_P}\,\in\,[m]^{[r-1]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k\in[r-1]:\ell_k=j} w_k}{c^j} \right) \right) \\
&= \sum_{j\in[m]} p_i^j \left(\frac{w_i}{c^j} + \sum_{k\in[n-1]\setminus[r-1]} \frac{p_k^j w_k}{c^j} + \sum_{\mathbf{L_P}\,\in\,[m]^{[r-1]}} \mathsf{Pr}_{\mathbf{P}}(\mathbf{L_P}) \left(\frac{\sum_{k\in[r-1]:\ell_k=j} w_k}{c^j} \right) \right),
\end{aligned}$$

as needed to prove the claim for r. Setting now $r = 0$ yields that

$$\lambda_i(\mathbf{P}) \;=\; \sum_{j\in[m]} p_i^j \, \frac{w_i + \sum_{k\in[n]:k\neq i} p_k^j w_k}{c^j}.$$

as needed. ■

For each user $i \in [n]$ and link $j \in [m]$, define the *Conditional Expected Individual Cost* [20] of user i on link j as

$$\lambda_i^j(\mathbf{P}) \;=\; \frac{w_i + \sum_{k\in[n]:k\neq i} p_k^j w_k}{c^j}.$$

By Lemma 3.1, $\lambda_i(\mathbf{P}) = \sum_{j\in[m]} p_i^j \lambda_i^j(\mathbf{P})$; thus, the definition of Nash equilibrium implies that for each link $j \in [m]$ such that $p_i^j > 0$, for each link $j' \in [m]$, either $\lambda_i^j = \lambda_i^{j'}$ if $p_i^{j'} > 0$ or $\lambda_i^j \leq \lambda_i^{j'}$ if $p_i^{j'} = 0$. (This property was used for the definition of Nash equilibrium in the original work of Koutsoupias and Papadimitriou [20].)

We now consider the case of identical users, for which we provide a characterization of optimal (pure) assignments.

Proposition 3.2 (Global Optimality = Local Optimality) *Consider a pure assignment* $\mathbf{Q}$ *for an instance* $\langle \mathbf{w}, \mathbf{c} \rangle$ *in the case of identical users. Then,* $\mathbf{Q}$ *is optimal if and only if for every pair of distinct links* $j, j' \in [m]$,

$$\frac{2\delta^j(\mathbf{Q}) + 1}{c^j} \;\geq\; \frac{2\delta^{j'}(\mathbf{Q}) - 1}{c^{j'}}.$$

Proof: Assume first that $\mathbf{Q}$ is optimal, so that $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \mathsf{QOPT}(\mathbf{w}, \mathbf{c})$. Consider any pair of distinct links $j, j' \in [m]$ and use $\mathbf{Q}$ to construct a new pure assignment $\mathbf{Q}'$ by switching any single user from link j' to link j. Then, clearly,

$$\begin{aligned}
\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}') - \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) &= \frac{(\delta^j(\mathbf{Q})+1)^2}{c^j} + \frac{(\delta^{j'}(\mathbf{Q})-1)^2}{c^{j'}} - \frac{(\delta^j(\mathbf{Q}))^2}{c^j} - \frac{(\delta^{j'}(\mathbf{Q}))^2}{c^{j'}} \\
&= \frac{2\delta^j(\mathbf{Q})+1}{c^j} - \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}}.
\end{aligned}$$

Since $\mathbf{Q}$ is optimal, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}') \geq \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q})$, so that

$$\frac{2\delta^{j}(\mathbf{Q})+1}{c^{j}} \geq \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}},$$

as needed.

Assume now that for every pair of links $j, j' \in [m]$,

$$\frac{2\delta^{j}(\mathbf{Q})+1}{c^{j}} \geq \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}}.$$

We will prove that $\mathbf{Q}$ is an optimal assignment for the instance $\langle \mathbf{w}, \mathbf{c} \rangle$. To do so, we will consider an optimal assignment $\mathbf{R}$ for the same instance, and we will prove that $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{R})$. If $\delta^{\ell}(\mathbf{Q}) = \delta^{\ell}(\mathbf{R})$ for all links $\ell \in [m]$, then $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{R})$ and the optimality of $\mathbf{Q}$ follows. So assume otherwise. Since $\sum_{\ell \in [m]} \delta^{\ell}(\mathbf{Q}) = \sum_{\ell \in [m]} \delta^{\ell}(\mathbf{R}) = n$, it follows that there exist distinct links $j, j' \in [m]$ such that $\delta^{j}(\mathbf{R}) > \delta^{j}(\mathbf{Q})$ and $\delta^{j'}(\mathbf{Q}) > \delta^{j'}(\mathbf{R})$; that is, link j is assigned more traffic in $\mathbf{R}$ than in $\mathbf{Q}$, while link j' is assigned more traffic in $\mathbf{Q}$ than in $\mathbf{R}$. Since loads are integral in the case of identical users, it follows that $\delta^{j}(\mathbf{R}) - \delta^{i}(\mathbf{Q}) \geq 1$ and $\delta^{j'}(\mathbf{Q}) - \delta^{j'}(\mathbf{R}) \geq 1$.

Since $\mathbf{R}$ is optimal, it holds for links $j', j \in [m]$ that

$$\frac{2\delta^{j'}(\mathbf{R})+1}{c^{j'}} \geq \frac{2\delta^{j}(\mathbf{R})-1}{c^{j}}.$$

On the other hand, our assumption on $\mathbf{Q}$ implies that

$$\frac{2\delta^{j}(\mathbf{Q})+1}{c^{j}} \geq \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}}.$$

It follows that

$$\begin{aligned}
& \frac{2\delta^{j'}(\mathbf{R})+1}{c^{j'}} \\
\geq\ & \tfrac{2\delta^{j}(\mathbf{R})-1}{c^{j}} \\
\geq\ & \tfrac{2\delta^{j}(\mathbf{Q})+1}{c^{j}} \quad (\text{since } \delta^{j}(\mathbf{R}) \geq \delta^{j}(\mathbf{Q})+1) \\
\geq\ & \tfrac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}} \\
\geq\ & \tfrac{2\delta^{j'}(\mathbf{R})+1}{c^{j'}} \quad (\text{since } \delta^{j'}(\mathbf{Q}) \geq \delta^{j'}(\mathbf{R})+1)
\end{aligned}$$

It follows that all terms in this sequence of inequalities are equal. In particular,

$$\frac{2\delta^{j}(\mathbf{R})-1}{c^{j}} = \frac{2\delta^{j}(\mathbf{Q})+1}{c^{j}},$$

or $\delta^j(\mathbf{R}) = \delta^j(\mathbf{Q}) + 1$; also,

$$\frac{2\delta^{j'}(\mathbf{Q}) - 1}{c^{j'}} = \frac{2\delta^{j'}(\mathbf{R}) + 1}{c^{j'}},$$

or $\delta^{j'}(\mathbf{Q}) = \delta^{j'}(\mathbf{R}) + 1$. Thus, the difference of the loads on the links $j, j' \in [m]$ is exactly 1 (in absolute value) in either $\mathbf{Q}$ or $\mathbf{R}$. Since $\sum_{l\in[m]} \delta^l(\mathbf{Q}) = \sum_{l\in[m]} \delta^l(\mathbf{R})$, this implies that the number of links $j \in [m]$ such that $\delta^j(\mathbf{R}) > \delta^j(\mathbf{Q})$ and the number of links $j' \in [m]$ such that $\delta^{j'}(\mathbf{Q}) > \delta^{j'}(\mathbf{R})$ are equal. Moreover, it also follows from the previous sequence of inequalities that

$$\frac{2\delta^{j'}(\mathbf{R}) + 1}{c^{j'}} = \frac{2\delta^{j}(\mathbf{Q}) + 1}{c^{j}},$$

which implies that

$$\frac{(\delta^j(\mathbf{Q}))^2}{c^j} + \frac{(\delta^{j'}(\mathbf{R}) + 1)^2}{c^{j'}} = \frac{(\delta^{j'}(\mathbf{R}))^2}{c^{j'}} + \frac{(\delta^j(\mathbf{Q}) + 1)^2}{c^j}.$$

Since $\delta^j(\mathbf{R}) = \delta^j(\mathbf{Q}) + 1$ and $\delta^{j'}(\mathbf{Q}) = \delta^{j'}(\mathbf{R}) + 1$, it follows that

$$\frac{(\delta^j(\mathbf{Q}))^2}{c^j} + \frac{(\delta^{j'}(\mathbf{Q}))^2}{c^{j'}} = \frac{(\delta^{j'}(\mathbf{R}))^2}{c^{j'}} + \frac{(\delta^j(\mathbf{R}))^2}{c^j}.$$

Thus, links $j, j' \in [m]$ have the same contribution to Quadratic Social Cost in both $\mathbf{Q}$ and $\mathbf{R}$. Since the number of links $j \in [m]$ such that $\delta^j(\mathbf{R}) > \delta^j(\mathbf{Q})$ and the number of links $j' \in [m]$ such that $\delta^{j'}(\mathbf{Q}) > \delta^{j'}(\mathbf{R})$ are equal, the total contribution to Quadratic Social Cost of links $j \in [m]$ such that $\delta^j(\mathbf{R}) > \delta^j(\mathbf{Q})$ is equal to the total contribution to Quadratic Social Cost of links $j' \in [m]$ such that $\delta^{j'}(\mathbf{Q}) > \delta^{j'}(\mathbf{R})$. It follows that $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{R})$, so that $\mathbf{Q}$ is optimal, as needed. ■

Proposition 3.2 establishes that the (global) optimality of a pure assignment is equivalent to a collection of $\Theta(m^2)$ *local* conditions – one for each pair of distinct links and each checkable in time $\Theta(n)$. Hence, an immediate consequence of Proposition 3.2 follows.

Corollary 3.3 *Consider the case of identical users. Then, the optimality of a pure assignment can be decided in time $O(n + m^2)$.*

We continue to prove a relation between pure Nash equilibria and optimal (pure) assignments in the case of identical users.

Lemma 3.4 (Optimal Assignment versus Nash Equilibrium) *Consider an optimal assignment $\mathbf{Q}$ and a pure Nash equilibrium $\mathbf{P}$ for an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ in the case of identical users. Then, for each link $j \in [m]$, $\delta^j(\mathbf{Q}) - \delta^j(\mathbf{P}) \leq 1$.*

Proof: Assume, by way of contradiction, that there is some link $j \in [m]$ such that $\delta^j(\mathbf{Q}) - \delta^j(\mathbf{P}) \geq 2$. Since $\sum_{l\in[m]} \delta^l(\mathbf{Q}) = \sum_{l\in[m]} \delta^l(\mathbf{P})$, there also exists some link $j' \in [m]$ such that $\delta^{j'}(\mathbf{Q}) - \delta^{j'}(\mathbf{P}) \leq -1$. Since $\mathbf{Q}$ is optimal, Lemma 3.2 implies that

$$\begin{array}{ll}
\multicolumn{2}{l}{\dfrac{2\delta^j(\mathbf{Q})+1}{c^j}} \\
\leq & \frac{2\delta^{j'}(\mathbf{Q})-1}{c^{j'}} \\
\leq & \frac{2(\delta^{j'}(\mathbf{P})-1)+1}{c^{j'}} \\
= & \frac{2\delta^{j}(\mathbf{P})-1}{c^{j}} \\
\leq & 2\,\frac{\delta^j(\mathbf{P})+1}{c^j} - \frac{1}{c^{j'}} \qquad \text{(since } \mathbf{P} \text{ is a Nash equilibrium)} \\
\leq & 2\,\frac{\delta^j(\mathbf{Q})-2+1}{c^j} - \frac{1}{c^{j'}} \qquad \text{(by assumption)} \\
= & \frac{2\delta^j(\mathbf{Q})-1}{c^j} - \frac{1}{c^j} - \frac{1}{c^{j'}}\,.
\end{array}$$

It follows that $\frac{1}{c^j} + \frac{1}{c^{j'}} \leq 0$, a contradiction. ∎

We now consider an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ for the case of identical links. Call user $i \in [n]$ *bursty* if $w_i > \frac{W}{m}$. Intuitively, the traffic of a bursty user exceeds the fair share of traffic for a link. We prove a simple property of bursty users:

Lemma 3.5 (Bursty Users are Solo) *Consider any instance in the case of identical links. Then, a bursty user is solo in either an optimal assignment or a pure Nash equilibrium.*

Proof: Fix an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and a bursty user $i \in [n]$.

Consider first an optimal assignment $\mathbf{Q} = \langle q_1, \ldots, q_n \rangle$. Note that $\delta^{q_i}(\mathbf{Q}) \geq w_i > \frac{W}{m}$; Since $\sum_{l\in[m]} \delta^l(\mathbf{Q}) = W$, there is some other link $j \in [m]$ with $j \neq q_i$ such that $\delta^j(\mathbf{Q}) < \frac{W}{m}$. Assume, by way of contradiction, that some user $k \neq i$ is assigned to link q_i. Modify $\mathbf{Q}$ to obtain $\mathbf{Q}'$ by switching user k to link j. Then,

$$\begin{array}{rcl}
\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}') - \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) & = & \left(\delta^{q_i}(\mathbf{Q}')\right)^2 + \left(\delta^j(\mathbf{Q}')\right)^2 - \left(\delta^{q_i}(\mathbf{Q})\right)^2 - \left(\delta^j(\mathbf{Q})\right)^2 \\
& = & w_i^2 + \left(\delta^j(\mathbf{Q}) + w_k\right)^2 - (w_i + w_k)^2 - \left(\delta^j(\mathbf{Q})\right)^2 \\
& = & 2w_k\left(\delta^j(\mathbf{Q}) - w_i\right).
\end{array}$$

Since $\delta^j(\mathbf{Q}) < \frac{W}{m}$ and $w_i > \frac{W}{m}$, it follows that $\delta^j(\mathbf{Q}) - w_i < 0$, so that $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}') < \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q})$. Since $\mathbf{Q}$ is optimal, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}') \geq \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q})$. A contradiction.

Consider now a pure Nash equilibrium $\mathbf{P} = \langle \ell_1, \ldots, \ell_n \rangle$. Note that $\delta^{\ell_i}(\mathbf{P}) \geq w_i > \frac{W}{m}$. Since $\sum_{l\in[m]} \delta^l(\mathbf{P}) = W$, there is some other link $j \in [m]$ with $j \neq \ell_i$ such that $\delta^j(\mathbf{P}) < \frac{W}{m}$. Assume, by way of contradiction, that some user $k \neq i$ is assigned to link ℓ_i. Then, $\lambda_k(\mathbf{P}) \geq w_i + w_k > \frac{W}{m} + w_k$. However, if user k switches to link j, its Individual Cost becomes $\delta^j(\mathbf{P}) + w_k < \frac{W}{m} + w_k$. Since $\mathbf{P}$ is a Nash equilibrium, $\delta^j(\mathbf{P}) + w_k \geq \lambda_k(\mathbf{P}) > \frac{W}{m} + w_k$. A contradiction. ∎

Say that an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ is *bursty* if some user $i \in [n]$ is bursty; else, the instance is *non-bursty*. We prove a preliminary property of non-bursty Nash equilibria for the case of identical links; roughly speaking, we prove that link loads are balanced in a non-bursty Nash equilibrium.

Lemma 3.6 (Nash Equilibrium Balances Loads in Non-Bursty Instance) *Consider a Nash equilibrium* $\mathbf{P}$ *for a non-bursty instance* $\langle \mathbf{w}, \mathbf{c} \rangle$ *in the case of identical links. Then, for each link* $j \in [m]$, $\delta^j(\mathbf{P}) \leq 2 \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P})$.

Proof: Assume, by way of contradiction, that there is some link $j \in [m]$ such that $\delta^j(\mathbf{P}) > 2 \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P})$. Take link j so that it maximizes $\delta^l(\mathbf{P})$ over all links $l \in [m]$.

Clearly, $\delta^j(\mathbf{P}) \geq \frac{W}{m}$. Moreover, if $\delta^j(\mathbf{P}) = \frac{W}{m}$, then $\delta^l(\mathbf{P}) = \frac{W}{m}$ for all links $l \in [m]$ and the claim follows. Hence, assume that $\delta^j(\mathbf{P}) > \frac{W}{m}$. We proceed by case analysis.

- Assume first that there is a single solo user $i \in [n]$ on link j, so that $\delta^j(\mathbf{P}) = w_i$. Since link j maximizes latency, $w_i \geq \delta^l(\mathbf{P})$ for all links $l \in [m]$. Moreover, our assumption implies that $w_i > \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P})$. It follows that $m w_i > \sum_{l \in [m]} \delta^l(\mathbf{P}) = W$, or $w_i > \frac{W}{m}$. Since $\langle \mathbf{w}, \mathbf{c} \rangle$ is a non-bursty instance, $w_i \leq \frac{W}{m}$. A contradiction.

- Assume now that at least two users are assigned to link j. Consider the smallest traffic w_i of some user $i \in [n]$ among all users assigned to link j. Then, clearly, $w_i \leq \frac{\delta^j(\mathbf{P})}{2}$. Hence, $\delta^j(\mathbf{P}) - w_i \geq \frac{\delta^j(\mathbf{P})}{2} > \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P})$ (by assumption). So, $\min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}) + w_i < \delta^j(\mathbf{P})$. Since $\mathbf{P}$ is a Nash equilibrium, $\min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}) + w_i \geq \delta^j(\mathbf{P})$. A contradiction.

Since we obtained a contradiction in all possible cases, the proof is now complete. ■

We finally discuss the issue of *link-minimality*. Consider a pair of an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and an associated pure Nash equilibrium $\mathbf{P}$. Say that the pair is *link-minimal* if it has the least number of links among all pairs of an instance $\langle \mathbf{w}', \mathbf{c}' \rangle$ and an associated Nash equilibrium $\mathbf{P}'$ such that $\frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QOPT}(\mathbf{w}', \mathbf{c}')} \geq \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$. We prove:

Lemma 3.7 (No Same Loads) *Consider a link-minimal pair of an instance* $\langle \mathbf{w}, \mathbf{c} \rangle$ *and an associated pure Nash equilibrium* $\mathbf{P}$. *Let* $\mathbf{Q}$ *be an optimal assignment for the instance* $\langle \mathbf{w}, \mathbf{c} \rangle$. *Then, there is no link* $j \in [m]$ *such that* $\delta^j(\mathbf{P}) = \delta^j(\mathbf{Q})$.

Proof: Assume, by way of contradiction, that there is such a link $j \in [m]$. We proceed as follows.

- Modify the instance $\langle \mathbf{w}, \mathbf{c} \rangle$ by eliminating link j and all users assigned to it. Denote $\langle \widehat{\mathbf{w}}, \widehat{\mathbf{c}} \rangle$ the resulting instance.
- Denote $\widehat{\mathbf{P}}$ and $\widehat{\mathbf{Q}}$ the pure assignments induced by this modification, which correspond to $\mathbf{P}$ and $\mathbf{Q}$, respectively.

Clearly,

$$\mathsf{QSC}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{P}}) = \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) - \frac{(\delta^j(\mathbf{Q}))^2}{c^j}$$

and

$$\mathsf{QOPT}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}) = \mathsf{QOPT}(\mathbf{w}, \mathbf{c}) - \frac{(\delta^j(\mathbf{Q}))^2}{c^j},$$

so that

$$\frac{\mathsf{QSC}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{P}})}{\mathsf{QOPT}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}})} = \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) - \frac{(\delta^j(\mathbf{Q}))^2}{c^j}}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) - \frac{(\delta^j(\mathbf{Q}))^2}{c^j}}.$$

Since the instance $\langle \widehat{\mathbf{w}}, \widehat{\mathbf{c}} \rangle$ has fewer links than the instance $\langle \mathbf{w}, \mathbf{c} \rangle$, and the pair of the instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and the pure Nash equilibrium $\mathbf{P}$ is link-minimal, it follows that

$$\frac{\mathsf{QSC}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{P}})}{\mathsf{QOPT}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}})} < \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})},$$

or

$$\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) - \frac{(\delta^j(\mathbf{Q}))^2}{c^j}}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) - \frac{(\delta^j(\mathbf{Q}))^2}{c^j}} < \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}.$$

Rearrangement of terms yields that $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) < \mathsf{QOPT}(\mathbf{w}, \mathbf{c})$. A contradiction. ■

4 Quadratic Social Cost

In this section, we study the Quadratic Social Cost of arbitrary (mixed) Nash equilibria. Some combinatorial expressions for Quadratic Social Cost, giving rise to corresponding efficient algorithms, are presented in Section 4.1. Section 4.2 determines the worst-case Nash equilibrium with respect to Quadratic Social Cost.

4.1 Combinatorial Expressions

We start by proving:

Proposition 4.1 (Quadratic Social Cost of Mixed Assignment) *The Quadratic Social Cost of any mixed assignment can be computed in time $O(nm)$.*

Proof: Fix any instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and a mixed assignment $\mathbf{P}$. Then,

$$
\begin{aligned}
& \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \\
= & \sum_{i \in [n]} w_i \lambda_i(\mathbf{P}) \\
= & \sum_{i \in [n]} w_i \sum_{j \in [m]} p_i^j \frac{w_i + \sum_{k \neq i} p_k^j w_k}{c^j} \qquad \text{(by Lemma 3.1)} \\
= & \sum_{i \in [n]} w_i \sum_{j \in [m]} p_i^j \frac{(1-p_i^j) w_i + \sum_{k \in [n]} p_k^j w_k}{c^j} \\
= & \sum_{i \in [n]} w_i \sum_{j \in [m]} p_i^j \frac{(1-p_i^j) w_i + \theta^j(\mathbf{P})}{c^j} .
\end{aligned}
$$

So, $\theta^j(\mathbf{P}) = \sum_{k \in [n]} p_k^j w_k$ is precomputed for all $j \in [m]$ in time $O(nm)$; then, the nested sum is computed in time $O(nm)$. Thus, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$ is computed in total time $O(nm)$, as needed. ■

Since a probability matrix $\mathbf{P}$ has size $O(nm)$, Proposition 4.1 implies that the Quadratic Social Cost of any mixed assignment can be computed in linear time. We remark that this achieved efficiency has not needed the assumption that the mixed assignment is a Nash equilibrium. We next establish that the Quadratic Social Cost takes a particularly nice form for the case of the fully mixed Nash equilibrium. We prove:

Theorem 4.2 (Quadratic Social Cost of Fully Mixed Nash Equilibrium) *Consider an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ in the case of identical links. Then,*

$$
\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) \;=\; W_1 + \frac{2}{m} W_2 .
$$

Proof: Recall that $f_i^j = \frac{1}{m}$ for all users $i \in [n]$ and links $j \in [m]$ in the fully mixed Nash equilibrium $\mathbf{F}$. Thus,

$$
\begin{aligned}
& \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) \\
= & \sum_{i \in [n]} w_i \, \lambda_i(\mathbf{F}) \\
= & \sum_{i \in [n]} w_i \sum_{\ell \in [m]} f_i^\ell \left(w_i + \sum_{k \in [n] : k \neq i} f_k^\ell w_k \right) \quad \text{(by Lemma 3.1)} \\
= & \sum_{i \in [n]} w_i \sum_{\ell \in [m]} \frac{1}{m} \left(w_i + \sum_{k \in [n] : k \neq i} \frac{1}{m} \, w_k \right) \\
= & \sum_{i \in [n]} w_i \left(w_i + \sum_{k \in [n] : k \neq i} \frac{1}{m} \, w_k \right) \\
= & W_1 + \frac{2}{m} W_2
\end{aligned}
$$

as needed. ■

Theorem 4.2 immediately implies:

Corollary 4.3 *Consider the case of identical links. Then, the Quadratic Social Cost of the fully mixed Nash Equilibrium can be computed in time $O(n^2)$.*

We continue to prove:

Theorem 4.4 *Consider an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ in the case of identical users for which the fully mixed Nash equilibrium $\mathbf{F}$ exists. Then,*

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = \frac{n(n+m-1)}{C}.$$

Proof: Recall that in the case of identical users, for all users $i \in [n]$ and each link $j \in [m]$, $f_i^j = \frac{m+n-1}{(n-1)C} c^j - \frac{1}{n-1}$. Thus,

$$\begin{aligned}
&\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) \\
= \;& \sum_{i \in [n]} \lambda_i(\mathbf{F}) \\
= \;& \sum_{i \in [n]} \left(\sum_{j \in [m]} \left(f_i^j \frac{1 + \sum_{k \in [n] : k \neq i} f_k^j}{c^j} \right) \right) \quad \text{(by Lemma 3.1)} \\
= \;& \sum_{i \in [n]} \left(\sum_{j \in [m]} \left(f_i^j \frac{1 + (n-1) f_i^j}{c^j} \right) \right) \\
= \;& n \sum_{j \in [m]} \left(f_i^j \frac{1 + (n-1) f_i^j}{c^j} \right) \\
= \;& n \sum_{j \in [m]} \left(f_i^j \frac{m+n-1}{C} \right) \\
= \;& \frac{n(m+n-1)}{C} \sum_{j \in [m]} f_i^j \\
= \;& \frac{n(m+n-1)}{C},
\end{aligned}$$

as needed. ■

Theorem 4.4 immediately implies:

Corollary 4.5 *Consider the case of identical users. Then, the Quadratic Social Cost of the fully mixed Nash Equilibrium can be computed in time $O(m)$.*

For the special case of identical users and identical links, Theorem 4.4 immediately implies:

Corollary 4.6 *Consider the case of identical users and identical links. Then,*

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = \frac{n(n+m-1)}{m}.$$

4.2 The Worst-Case Nash Equilibrium

We restrict attention to the case of identical users and identical links, for which the fully mixed Nash equilibrium always exists. We start by proving:

Lemma 4.7 *Consider the case of identical users and identical links, and let* $\mathbf{P}$ *be any (mixed) Nash equilibrium. Then, for all users* $i \in [n]$, $\lambda_i(\mathbf{P}) \leq \lambda_i(\mathbf{F})$.

Proof: Assume, by way of contradiction, that $\lambda_i(\mathbf{P}) > \lambda_i(\mathbf{F})$, for some user $i \in [n]$.

For the Nash equilibrium $\mathbf{P}$, recall that all Conditional Expected Individual Costs λ_i^j for those links such that $p_i^j > 0$ are equal; hence, Lemma 3.1 implies that $\lambda_i(\mathbf{P}) = \lambda_i^j(\mathbf{P}) = 1 + \sum_{k \in [n]: k \neq i} p_k^j$. Moreover, all Conditional Expected Individual Costs for those links $j \in [m]$ such that $p_i^j = 0$ are no less than $\lambda_i(\mathbf{P})$. For the fully mixed Nash equilibrium $\mathbf{F}$, the same argument implies that $\lambda_i(\mathbf{F}) = \lambda_i^j(\mathbf{F})$ for any link $j \in [m]$.

By Lemma 3.1, $\lambda_i^j(\mathbf{P}) = 1 + \sum_{k \in [n]: k \neq i} p_k^j$ and $\lambda_i^j(\mathbf{F}) = 1 + \sum_{k \in [n]: k \neq i} f_k^j$. Since $\lambda_i(\mathbf{P}) > \lambda_i(\mathbf{F})$, it follows that $\sum_{k \in [n]: k \neq i} p_k^j > \sum_{k \in [n]: k \neq i} p_k^j$. However,

$$\sum_{\ell \in [m]} \left(\sum_{k \in [n]: k \neq i} p_k^\ell \right) = \sum_{\ell \in [m]} \left(\sum_{k \in [n]: k \neq i} f_k^\ell \right) = n - 1 \, .$$

It follows that there exists some link $j' \in [m]$ such that $\sum_{k \in [n]: k \neq i} p_k^{j'} < \sum_{k \in [n]: k \neq i} p_k^{j'}$. By definition of Conditional Expected Individual Cost, this implies that $\lambda_i^{j'}(\mathbf{P}) < \lambda_i^{j'}(\mathbf{F})$. However, $\lambda_i(\mathbf{P}) \leq \lambda_i^{j'}(\mathbf{P})$, while $\lambda_i(\mathbf{F}) = \lambda_i^{j'}(\mathbf{F})$. It follows that $\lambda_i(\mathbf{P}) < \lambda_i(\mathbf{F})$, a contradiction. ∎

Since the Quadratic Social Cost is a weighted sum of Expected Individual Costs, Lemma 4.7 immediately implies:

Theorem 4.8 *Consider the case of identical users and identical links. Then, for any arbitrary Nash equilibrium* $\mathbf{P}$, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \leq \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})$.

Theorem 4.8 proves that the fully mixed Nash equilibrium maximizes Quadratic Social Cost in the case of identical users and identical links. It is natural to ask if the same holds in all cases. In analogy to the yet unproven *Fully Mixed Nash Equilibrium Conjecture* [18], we conjecture:

Conjecture 4.1 (Quadratic Fully Mixed Nash Equilibrium Conjecture) *When it exists, the fully mixed Nash equilibrium maximizes Quadratic Social Cost.*

5 Quadratic Coordination Ratio

In this section, we present our bounds on Quadratic Coordination Ratio. Some of our proofs will make use of the following notation. Consider an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ with an associated pure assignment $\mathbf{P}$. Fix a set of links $\mathcal{L}$, inducing a set of users $\mathcal{U}$ that are assigned by the assignment $\mathbf{P}$ to links in $\mathcal{L}$. Then, $\mathbf{w} \setminus \mathcal{U}$ and $\mathbf{c} \setminus \mathcal{L}$ denote the vectors resulting from $\mathbf{w}$ and $\mathbf{c}$, respectively, by eliminating the entries corresponding to users in $\mathcal{U}$ and links in $\mathcal{L}$, respectively; $\mathbf{P} \setminus (\mathcal{U}, \mathcal{L})$ denotes the assignment induced by these eliminations. Pure and mixed Nash equilibria are considered in Sections 5.1 and 5.2,

5.1 Pure Nash Equilibria

Identical users and identical links are considered in Sections 5.1.1 and 5.1.2, respectively.

5.1.1 Identical Users

As our main result, we prove:

Theorem 5.1 (Quadratic Coordination Ratio for Pure Nash Equilibria) *Consider the case of identical users, restricted to pure Nash equilibria. Then,* $\mathsf{QCR} = \frac{4}{3}$.

Proof: We first prove the upper bound. Consider any pair of an arbitrary instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and an associated pure Nash equilibrium $\mathbf{P}$. Assume, without loss of generality, that the pair is link-minimal. (If it were not, replace it with a link-minimal pair achieving a ratio of Quadratic Social Cost over Quadratic Optimum no less than $\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$.) Let $\mathbf{Q}$ be an optimal (pure) assignment for the instance $\langle \mathbf{w}, \mathbf{c} \rangle$; so, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = \mathsf{QOPT}(\mathbf{w}, \mathbf{c})$. We start by proving a technical fact.

Lemma 5.2 *There is some link $j \in [m]$ such that for each link $l \in [m]$,*

$$\delta^l(\mathbf{P}) \quad = \quad \begin{cases} \delta^l(\mathbf{Q}) + m - 1, & \textit{if } l = j \\ \delta^l(\mathbf{Q}) - 1, & \textit{if } l \in [m] \setminus \{j\} \end{cases} .$$

Proof: Lemma 3.7 implies that there is a link $j \in [m]$ such that $\delta^j(\mathbf{P}) > \delta^j(\mathbf{Q})$. Since $\sum_{l \in [m]} \delta^l(\mathbf{P}) = \sum_{l \in [m]} \delta^l(\mathbf{Q})$, it follows that there is also some link $j' \in [m]$ such that $\delta^{j'}(\mathbf{P}) < \delta^{j'}(\mathbf{Q})$. Lemma 3.4 implies that $\delta^{j'}(\mathbf{P}) \geq \delta^{j'}(\mathbf{Q}) - 1$. Hence, $\delta^{j'}(\mathbf{P}) = \delta^{j'}(\mathbf{Q}) - 1$. It follows that there are at least $\delta^j(\mathbf{P}) - \delta^j(\mathbf{Q})$ such links $j' \in [m]$. Fix $\mathcal{L}$ to be a set of $\delta^j(\mathbf{P}) - \delta^j(\mathbf{Q})$

links $j' \in [m]$, such that $\delta^{j'}(\mathbf{P}) = \delta^{j'}(\mathbf{Q}) - 1$. Clearly,

$$\begin{aligned}
\sum_{l \in \mathcal{L} \cup \{j\}} \delta^l(\mathbf{P}) &= \sum_{l \in calL} \delta^l(\mathbf{P}) + \delta^j(\mathbf{P}) \\
&= \sum_{l \in \mathcal{L}} \Big(\delta^l(\mathbf{Q}) - 1 \Big) + \delta^l(\mathbf{Q}) + |\mathcal{L}| \\
&= \sum_{l \in calL} \delta^l(\mathbf{Q})\,.
\end{aligned}$$

Since users are identical, it follows that $\mathbf{P}$ and $\mathbf{Q}$ assign the same total number of users to links from the set $\mathcal{L} \cup \{j\}$. This implies that $\mathbf{P}$ and $\mathbf{Q}$ also assign the same total number of users to links from the set $[m] \setminus (\mathcal{L} \cup \{j\})$.

- So, denote $\langle \mathbf{w}', \mathbf{c}'$ the instance obtained by eliminating all links $l \in \mathcal{L} \cup \{j\}$ and all users assigned to them (by either $\mathbf{P}$ or $\mathbf{Q}$). Clearly, $|\mathbf{w}'| = \sum_{l \in [m] \setminus (\mathcal{L} \cup \{j\})} \delta^l(\mathbf{P}) = \sum_{l \in [m] \setminus (\mathcal{L} \cup \{j\})} \delta^l(\mathbf{Q})$.
- Denote also $\langle \mathbf{w}'', \mathbf{c}''$ the instance obtained by eliminating all links $l \in [m] \setminus (\mathcal{L} \cup \{j\})$ and all users assigned to them (by either $\mathbf{P}$ or $\mathbf{Q}$). Clearly, $|\mathbf{w}''| = \sum_{l \in \mathcal{L} \cup \{j\}} \delta^l(\mathbf{P}) = \sum_{l \in \mathcal{L} \cup \{j\}} \delta^l(\mathbf{Q})$.

Denote now

$$\begin{aligned}
A &= \textstyle\sum_{l \in [m] \setminus (\mathcal{L} \cup \{j\})} \dfrac{(\delta^l(\mathbf{P}))^2}{c^l}, & B &= \textstyle\sum_{l \in [m] \setminus (\mathcal{L} \cup \{j\})} \dfrac{(\delta^l(\mathbf{Q}))^2}{c^l}, \\
C &= \textstyle\sum_{l \in \mathcal{L} \cup \{j\}} \dfrac{(\delta^l(\mathbf{P}))^2}{c^l}, & D &= \textstyle\sum_{l \in \mathcal{L} \cup \{j\}} \dfrac{(\delta^l(\mathbf{Q}))^2}{c^l}\,.
\end{aligned}$$

Since $\delta^j(\mathbf{P}) > \delta^j(\mathbf{Q})$, it follows that $\delta^j(\mathbf{P}) > 0$. This implies that both $C > 0$ and $D > 0$.

We now prove that there is no other link $j' \in [m]$, $j' \neq j$, such that $\delta^{j'}(\mathbf{P}) > \delta^{j'}(\mathbf{Q})$. By way of contradiction, assume otherwise. By definition of $\mathcal{L}$, $j' \notin \mathcal{L}$. It follows that $j' \in [m] \setminus (\mathcal{L} \cup \{j\})$. Since $\delta^{j'}(\mathbf{P}) > \delta^{j'}(\mathbf{Q})$, it follows that $\delta^{j'}(\mathbf{P}) > 0$. Hence, both $A > 0$ and $B > 0$ as well.

Use now the instances $\langle \mathbf{w}', \mathbf{c}' \rangle$ and $\langle \mathbf{w}'', \mathbf{c}'' \rangle$ and the pure assignments $\mathbf{P}$ and $\mathbf{Q}$ to define four new pairs of instances and associated pure assignments, as follows:

- Instance $\langle \mathbf{w}', \mathbf{c}' \rangle$ and pure assignment $\mathbf{P}'$ (resp., pure assignment $\mathbf{Q}'$):
  – Eliminate all links $l \in \mathcal{L} \cup \{j\}$ and the users assigned to them; restrict $\mathbf{P}$ (resp., $\mathbf{Q}$) to the remaining links and users.
- Instance $\langle \mathbf{w}'', \mathbf{c}'' \rangle$ and pure assignment $\mathbf{P}''$ (resp., pure assignment $\mathbf{Q}''$):
  – Eliminate all links $l \in [m] \setminus (\mathcal{L} \cup \{j\})$ and the users assigned to them; restrict $\mathbf{P}$ (resp., $\mathbf{Q}$) to the ramaining links and users.

We continue to argue that $\mathbf{P}'$ and $\mathbf{P}''$ are Nash equilibria for the resulting instances $\langle \mathbf{w}', \mathbf{c}' \rangle$ and $\langle \mathbf{w}'', \mathbf{c}'' \rangle$, respectively; also, that $\mathbf{Q}'$ and $\mathbf{Q}''$ are optimal assignments for the resulting instances $\langle \mathbf{w}', \mathbf{c}' \rangle$ and $\langle \mathbf{w}'', \mathbf{c}'' \rangle$, respectively;

- Recall that $\mathbf{P}$ is a Nash equilibrium for the instance $\langle \mathbf{w}, \mathbf{c} \rangle$, while $\mathbf{P}'$ and $\mathbf{P}''$ result by deleting links and their assigned users. It follows that $\mathbf{P}'$ and $\mathbf{P}''$ are Nash equilibria for the resulting instances $\langle \mathbf{w}', \mathbf{c}' \rangle$ and $\langle \mathbf{w}'', \mathbf{c}'' \rangle$, respectively.

- By Lemma 3.2, the optimality of the pure assignment $\mathbf{Q}$ is equivalent to a set of conditions on the loads induced by $\mathbf{Q}$. Since the pure assignment $\mathbf{Q}'$ (resp., $\mathbf{Q}''$) preserves the set of loads, it follows that this set of conditions still holds on the set of loads induced by $\mathbf{Q}'$ (resp., $\mathbf{Q}''$). By Lemma 3.2, it follows that $\mathbf{Q}'$ (resp., $\mathbf{Q}''$) is an optimal assignment for the instance $\langle \mathbf{w}', \mathbf{c}' \rangle$ (resp., $\langle \mathbf{w}'', \mathbf{c}'' \rangle$).

Note that, by construction, $\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}') = A$, $\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{Q}') = B$, $\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'') = C$ and $\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{Q}'') = D$. Since $\mathbf{Q}'$ and $\mathbf{Q}''$ are optimal pure assignments for the instances $\langle \mathbf{w}', \mathbf{c}' \rangle$ and $\langle \mathbf{w}'', \mathbf{c}'' \rangle$, respectively, it follows that $\mathsf{QOPT}(\mathbf{w}', \mathbf{c}') = B$ and $\mathsf{QOPT}(\mathbf{w}'', \mathbf{c}'') = D$. We are interested in the ratios $\frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QOPT}(\mathbf{w}', \mathbf{c}')} = \frac{A}{B}$ and $\frac{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'')}{\mathsf{QOPT}(\mathbf{w}'', \mathbf{c}'')} = \frac{C}{D}$. In particular, we will compare these two ratios to the ratio $\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$. We proceed by case analysis on how the two ratios compare to each other.

- Assume first that $\frac{A}{B} > \frac{C}{D}$. It follows that $\frac{A}{B} > \frac{A+C}{B+D}$, or $\frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QOPT}(\mathbf{w}', \mathbf{c}')} > \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$. Since the instance $\langle \mathbf{w}', \mathbf{c}' \rangle$ has fewer links than the instance $\langle \mathbf{w}, \mathbf{c} \rangle$, this implies that the pair of the instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and the Nash equilibrium $\mathbf{P}$ is not link-minimal. A contradiction.

- Assume now that $\frac{C}{D} \geq \frac{A}{B}$. It follows that $\frac{C}{D} \geq \frac{A+C}{B+D}$, or $\frac{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'')}{\mathsf{QOPT}(\mathbf{w}'', \mathbf{c}'')} \geq \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$. Since the instance $\langle \mathbf{w}'', \mathbf{c}'' \rangle$ has fewer links than the instance $\langle \mathbf{w}, \mathbf{c} \rangle$, this implies that the pair of the instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and the Nash equilibrium $\mathbf{P}$ is not link-minimal. A contradiction.

Hence, it follows that there is no link $j' \in [m]$, $j' \neq j$, such that $\delta^{j'}(\mathbf{P}) > \delta^{j'}(\mathbf{Q})$. So, necessarily, Recall that $\delta^{j'}(\mathbf{P}) = \delta^{j'}(\mathbf{Q}) - 1$ for any link $j' \in \mathcal{L}$. Since $\sum_{l \in [m]} \delta^{l}(\mathbf{P}) = \sum_{l \in [m]} \delta^{l}(\mathbf{Q})$, this implies that $\delta^{j}(\mathbf{P}) = \delta^{j}(\mathbf{Q}) + m - 1$, and the claim follows. ∎

We continue to prove:

Lemma 5.3 *There is an instance $\langle \widetilde{\mathbf{w}}, \widetilde{\mathbf{c}} \rangle$ with the same number of links as $\langle \mathbf{w}, \mathbf{c} \rangle$ and with associated pure Nash equilibrium $\widetilde{\mathbf{P}}$ and optimal assignment $\widetilde{\mathbf{Q}}$ such that:*

1. There is a link $j \in [m]$ such that:

(a) $\delta^j(\widetilde{\mathbf{P}}) = \delta^j(\widetilde{\mathbf{Q}}) + m - 1$.

(b) For each pair of distinct links $l, l' \in [m] \setminus \{j\}$:

i. $\delta^l(\widetilde{\mathbf{Q}}) = \delta^{l'}(\widetilde{\mathbf{Q}})$.

ii. $\widetilde{c}^l = \widetilde{c}^{l'}$.

2. $\dfrac{\mathsf{QSC}(\widetilde{\mathbf{w}}, \widetilde{\mathbf{c}}, \widetilde{\mathbf{P}})}{\mathsf{QOPT}(\widetilde{\mathbf{w}}, \widetilde{\mathbf{c}})} \geq \dfrac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$.

Proof: By Lemma 5.2, there exists a (unique) link $j \in [m]$ such that $\delta^j(\mathbf{P}) = \delta^j(\mathbf{Q}) + m - 1$. We will prove Condition *1* for link j. Denote $\mathcal{L} \subsetneq [m] \setminus \{j\}$ be a largest set of links that have identical capacities and identical loads in $\mathbf{Q}$; that is, for all links $l, l' \in \mathcal{L}$, $c^l = c^{l'}$ and $\delta^l(\mathbf{Q}) = \delta^{l'}(\mathbf{Q})$. So, for any link $l \in \mathcal{L}$, denote $X = \dfrac{(\delta^l(\mathbf{Q}) - 1)^2}{c^l}$ and $Y = \dfrac{(\delta^l(\mathbf{Q}))^2}{c^l}$.

Fix any link $j' \in [m] \setminus (\mathcal{L} \cup \{j\})$. Denote:

$$\begin{array}{rclrcl} A & = & \sum_{l \in [m] \setminus (\mathcal{L} \cup \{j'\})} \dfrac{(\delta^j(\mathbf{P}))^2}{c^l}, & B & = & \sum_{l \in [m] \setminus (\mathcal{L} \cup \{j'\})} \dfrac{(\delta^l(\mathbf{Q}))^2}{c^l}, \\ C & = & \dfrac{(\delta^{j'}(\mathbf{Q}) - 1)^2}{c^{j'}}, & D & = & \dfrac{(\delta^{j'}(\mathbf{Q}))^2}{c^{j'}}. \end{array}$$

Then, clearly, $\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) = \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) = B + D + |\mathcal{L}|Y$ and $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = A + C + |\mathcal{L}|X$.

Modify now the instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and the pure assignment $\mathbf{P}$ (resp., pure assignment $\mathbf{Q}$) to obtain two new instances $\langle \mathbf{w}', \mathbf{c}' \rangle$ and $\langle \mathbf{w}'', \mathbf{c}'' \rangle$ and two corresponding new pure assignments $\mathbf{P}'$ and $\mathbf{P}''$ (resp., pure assignments $\mathbf{Q}'$ and $\mathbf{Q}''$) as follows:

- Instance $\langle \mathbf{w}', \mathbf{c}' \rangle$ and pure assignment $\mathbf{P}'$ (resp., pure assignment $\mathbf{Q}'$):
  – Set $c^{j'} := c^l$ for any link $l \in \mathcal{L}$
  – Set $\delta^{j'}(\mathbf{P}') := \delta^l(\mathbf{P})$ (resp., set $\delta^{j'}(\mathbf{Q}') := \delta^l(\mathbf{Q})$) for any link $l \in \mathcal{L}$
  Roughly speaking, the capacity and load of link j' are set to those of any link l from $\mathcal{L}$.

- Instance $\langle \mathbf{w}'', \mathbf{c}'' \rangle$ and pure assignment $\mathbf{P}''$ (resp., pure assignment $\mathbf{Q}''$):
  – For each link $l \in \mathcal{L}$, set $c^l := c^{j'}$
  – For each link $l \in \mathcal{L}$, set $\delta^l(\mathbf{P}'') := \delta^i(\mathbf{P})$ (resp., set $\delta^l(\mathbf{Q}'') := \delta^{j'}(\mathbf{Q})$) for any link $l \in \mathcal{L}$
  Roughly speaking, the capacity and load of each link l from $\mathcal{L}$ are set to those of link j'.

We make some simple claims about the new instances and pure assignments:

- Note that, by construction, the pair of the instance $\langle \mathbf{w}', \mathbf{c}' \rangle$ and the pure assignment $\mathbf{P}'$ (resp., the pair of the instance $\langle \mathbf{w}'', \mathbf{c}'' \rangle$ and the pure assignment $\mathbf{P}''$) preserve the set of link capacities and link loads in the instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and the awsociated Nash equilibrium $\mathbf{P}$. Since a Nash equilibrium is defined by a set of conditions on link loads and link capacities, it follows that $\mathbf{P}'$ and $\mathbf{P}''$ are Nash equilibria for the instances $\langle \mathbf{w}', \mathbf{c}' \rangle$ and $\langle \mathbf{w}'', \mathbf{c}'' \rangle$, respectively.

- By Lemma 3.2, the optimality of the pure assignment $\mathbf{Q}$ is equivalent to a set of conditions on the set of link loads induced by $\mathbf{Q}$. Since the pure assignment $\mathbf{Q}'$ (resp., $\mathbf{Q}''$) preserves the set of link loads, it follows that the same set of conditions still holds on the set of link loads induced by $\mathbf{Q}'$ (resp., $\mathbf{Q}''$). By Lemma 3.2, it follows that $\mathbf{Q}'$ (resp., $\mathbf{Q}''$) is an optimal assignment for the instance $\langle \mathbf{w}', \mathbf{c}' \rangle$ (resp., $\langle \mathbf{w}'', \mathbf{c}'' \rangle$).

Note also that, by construction,

$$\begin{aligned}
\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}') &= A + (|\mathcal{L}| + 1)X\,, \\
\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{Q}') &= B + (|\mathcal{L}| + 1)Y\,, \\
\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'') &= A + (|\mathcal{L}| + 1)C
\end{aligned}$$

and

$$\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{Q}'') = B + (|\mathcal{L}| + 1)D\,.$$

We are interested in the ratios $\frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{Q}')}$ and $\frac{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'')}{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{Q}'')}$. We now show that at least one of these ratios exceeds the original ratio $\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$.

Claim 5.4 *It holds that*

$$\max\left\{ \frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{Q}')}, \frac{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'')}{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{Q}'')} \right\} \geq \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}\,.$$

Proof: Assume, by way of contradiction, that both $\frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{Q}')} < \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$ and $\frac{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'')}{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{Q}'')} < \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$. So, in our notation, both

$$(A + (|\mathcal{L}| + 1)X)(B + D + |\mathcal{L}|Y) < (A + V + |\mathcal{L}|X)(B + (|\mathcal{L}| + 1)Y)$$

and

$$(A + (|\mathcal{L}| + 1)V)(B + D + |\mathcal{L}|Y) < (A + C + |\mathcal{L}|X)(B + (|\mathcal{L}| + 1)D)\,.$$

These are equivalent to

$$AD + BX + (|\mathcal{L}| + 1)DX \quad < \quad AY + BC + (|\mathcal{L}| + 1)CY$$

and

$$AY + BC + (|\mathcal{L}| + 1)CY \quad < \quad AD + BX + (|\mathcal{L}| + 1)DX\,.$$

A contradiction. ■

Take now $\langle \widehat{\mathbf{w}}, \widehat{\mathbf{c}} \rangle$ and $\widehat{\mathbf{P}}$ to be the one between $\langle \mathbf{w}', \mathbf{c}' \rangle$ and $\mathbf{P}'$, and $\langle \mathbf{w}'', \mathbf{c}'' \rangle$ and $\mathbf{P}''$ that attains the maximum ratio between $\frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{Q}')}$ and $\frac{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'')}{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{Q}'')}$.

By our choice, $\langle \widehat{\mathbf{w}}, \widehat{\mathbf{c}} \rangle$ has the same number of links as the original instance $\langle \mathbf{w}, \mathbf{c} \rangle$ (since each of $\langle \mathbf{w}', \mathbf{c}' \rangle$ and $\langle \mathbf{w}'', \mathbf{c}'' \rangle$ does). Since both $\mathbf{P}'$ and $\mathbf{P}''$ are Nash equilibria for the instances $\langle \mathbf{w}', \mathbf{c}' \rangle$ and $\langle \mathbf{w}'', \mathbf{c}'' \rangle$, respectively, it follows that $\widehat{\mathbf{P}}$ is a Nash equilibrium for the instance $\langle \widehat{\mathbf{w}}, \widehat{\mathbf{c}} \rangle$. We will show that this choice satisfies all conditions in the claim.

- Claim 5.4 implies that $\max\left\{ \frac{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{P}')}{\mathsf{QSC}(\mathbf{w}', \mathbf{c}', \mathbf{Q}')}, \frac{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{P}'')}{\mathsf{QSC}(\mathbf{w}'', \mathbf{c}'', \mathbf{Q}'')} \right\} \geq \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$, so that Condition *2* holds.
- Now, Condition *1/(a)* holds since link j and all users assigned to it have not been affected by our construction, while the condition held for the original instance (by Lemma 5.2).

  Moreover, our construction has increased by one the set of links that have same capacities and same loads. Thus, Condition *1/(b)* follows inductively.

Our proof is now complete. ■

Set now $\delta^1(\mathbf{Q}) := \widehat{\delta}$, $c^1 := c$ and $\delta^2(\mathbf{Q}) := \delta$. By Lemma 5.2, $\delta^1(\mathbf{P}) = \widehat{\delta} + m - 1$ and $\delta^2(\mathbf{P}) = \delta - 1$. By Lemma 5.3, it follows that $c^l = 1$ and $\delta^l(\mathbf{Q}) = \delta^2(\mathbf{Q}) = \delta$ for all links $l \in [m] \setminus \{1\}$. By Lemma 5.2, this implies that $\delta^l(\mathbf{P}) = \delta - 1$ for all links $l \in [m] \setminus \{1\}$. Since $\delta^l(\mathbf{P}) \geq 0$ for all links $l \in [m]$, it follows that $\delta \geq 1$. Thus,

$$\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \quad = \quad \frac{(\widehat{\delta}(\mathbf{Q}) + m - 1)^2}{c} + (m-1)(\delta - 1)^2$$

and

$$\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) \quad = \quad \frac{(\widehat{\delta}(\mathbf{Q}))^2}{c} + (m-1)\delta^2\,.$$

Thus, the ratio $\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})}$ is bounded from above by the value of the following optimization problem Π_1 over the three variables $\widehat{\delta}$, δ and c:

$$\Pi_1 : \quad \underset{\widehat{\delta}\in \mathrm{N}_0, \delta\in \mathrm{N}, c\in \mathrm{R}_{>0}}{\text{Maximize}} \quad \frac{\frac{(\widehat{\delta}+m-1)^2}{c} + (m-1)(\delta-1)^2}{\frac{\widehat{\delta}^2}{c} + (m-1)\delta^2}$$

subject to:
(1/a) $\frac{2\widehat{\delta}+1}{c} \geq 2\delta - 1$ (1/b) $2\delta + 1 \geq \frac{2\widehat{\delta}-1}{c}$
(2/a) $\frac{\widehat{\delta}+m-1}{c} \leq \delta$ (2/b) $\delta - 1 \leq \frac{\widehat{\delta}+m}{c}$

Conditions (1/a) and (1/b) are necessary and sufficient conditions for the pure assignment $\mathbf{Q}$ to be optimal (by Proposition 3.2). Moreover, Conditions (2/a) and (2/b) are necessary and sufficient conditions for the pure assignment $\mathbf{P}$ to be a Nash equilibrium. We observe:

- Condition (1/a) implies Condition (2/b). So, Condition (2/b) can be dropped.
- Clearly, $\frac{(\widehat{\delta}+m-1)^2}{c} > \frac{\widehat{\delta}^2}{c}$ while $(m-1)(\delta-1)^2 < (m-1)\delta^2$. Thus, to maximize the function $\frac{(\widehat{\delta}+1)^2+c(m-1)(\delta-1)^2}{\widehat{\delta}^2+c(m-1)\delta^2}$, it is necessary to minimize c.

  By Condition (1/b), $c \geq \frac{2\widehat{\delta}-1}{2\delta+1}$; by Condition (2/a), $c \geq \frac{\widehat{\delta}+m-1}{\delta}$. However, $\frac{\widehat{\delta}+m-1}{\delta} > \frac{2\widehat{\delta}+2(m-1)}{2\delta+1} > \frac{2\widehat{\delta}-1}{2\delta+1}$ (since $m \geq 2$). So, set $c = \frac{\widehat{\delta}+m-1}{\delta}$ and drop Conditions (1/b) and (2/a).
- Condition (1/a) is now the only remaining condition; since, $c = \frac{\widehat{\delta}+m-1}{\delta}$, it becomes $\delta(2\widehat{\delta}+1) \geq (2\delta-1)(\widehat{\delta}+m-1)$ or $\widehat{\delta}(2\delta-(2\delta-1)) \geq (2\delta-1)(m-1)-\delta$ or $\widehat{\delta} \geq \delta(2m-3)-(m-1)$.

Setting $c = \frac{\widehat{\delta}+m-1}{\delta}$, the function to be maximized becomes

$$\begin{aligned}
& \frac{\frac{(\widehat{\delta}+m-1)^2\delta}{\widehat{\delta}+m-1} + (m-1)(\delta-1)^2}{\frac{\widehat{\delta}^2\delta}{\widehat{\delta}+m-1} + (m-1)\delta^2} \\
= \; & \frac{(\widehat{\delta}+m-1)^2\delta + (m-1)(\delta-1)^2(\widehat{\delta}+m-1)}{\widehat{\delta}^2\delta + (m-1)\delta^2(\widehat{\delta}+m-1)} \\
= \; & 1 + \frac{(m-1)^2\delta + 2\widehat{\delta}(m-1)\delta + (m-1)(-2\delta+1)(\widehat{\delta}+m-1)}{\widehat{\delta}^2\delta + (m-1)\delta^2(\widehat{\delta}+m-1)} \\
= \; & 1 + \frac{(m-1)^2\delta + 2\widehat{\delta}(m-1)\delta + (m-1)(\widehat{\delta}+m-1) - 2\delta(m-1)(\widehat{\delta}+m-1)}{\widehat{\delta}^2\delta + (m-1)\delta^2(\widehat{\delta}+m-1)} \\
= \; & 1 + \frac{(m-1)^2\delta + (m-1)(\widehat{\delta}+m-1) - 2\delta(m-1)^2}{\widehat{\delta}^2\delta + (m-1)\delta^2(\widehat{\delta}+m-1)} \\
= \; & 1 + \frac{-(m-1)^2\delta + (m-1)(\widehat{\delta}+m-1)}{\widehat{\delta}^2\delta + (m-1)\delta^2(\widehat{\delta}+m-1)} .
\end{aligned}$$

Thus, our optimization problem Π_1 now simplifies to the optimization problem Π_2:

$$\Pi_2: \quad \underset{\widehat{\delta}\in \mathrm{N}_0, \delta \in \mathrm{N}}{\text{Maximize}} \quad 1 + \frac{-(m-1)^2 + \frac{(m-1)(\widehat{\delta}+m-1)}{\delta}}{\widehat{\delta}^2 + (m-1)\delta(\widehat{\delta}+m-1)}$$

subject to: **(1/a)** $\widehat{\delta} \geq \delta(2m-3) - (m-1)$

Clearly, the numerator in the fraction to be maximized is monotonically decreasing in δ, while the denominator in the fraction is monotonically increasing in δ. Since $\delta \geq 1$, the numerator is maximized for $\delta = 1$, while the denominator is minimized for $\delta = 1$. It follows that the fraction is maximized for $\delta = 1$. This observation implies the following further simplification to our optimization problem Π_2:

$$\Pi_3: \quad \underset{\widehat{\delta}\in \mathrm{N}_0}{\text{Maximize}} \quad 1 + \frac{(m-1)\widehat{\delta}}{\widehat{\delta}^2 + (m-1)(\widehat{\delta}+m-1)}$$

subject to: **(1/a)** $\widehat{\delta} \geq m-2$.

We proceed to determine the value of the optimization problem Π_3. Define the real function

$$f(\hat{\delta}) \quad = \quad \frac{(m-1)\widehat{\delta}}{\widehat{\delta}^2 + (m-1)(\widehat{\delta}+m-1)}$$

of a real variable $\widehat{\delta}$. (Here, m is a fixed constant.) Clearly, $\mathsf{QCR} \leq \sup_{\widehat{\delta}} f(\widehat{\delta})$. To maximize the function $f(\widehat{\delta})$, observe that the first derivative of $f(\widehat{\delta})$ is

$$\begin{aligned}
\frac{df(\widehat{\delta})}{d\widehat{\delta}} &= \frac{\left(\widehat{\delta}^2 + (m-1)(\widehat{\delta}+m-1)\right)(m-1) - (m-1)\widehat{\delta}(2\widehat{\delta}+m-1)}{\left(\widehat{\delta}^2 + (m-1)(\widehat{\delta}+m-1)\right)^2} \\
&= \frac{-(m-1)\widehat{\delta}^2 + (m-1)^3 -}{\left(\widehat{\delta}^2 + (m-1)(\widehat{\delta}+m-1)\right)^2} \\
&= \frac{(m-1)\left((m-1)^2 - \widehat{\delta}^2\right)}{\left(\widehat{\delta}^2 + (m-1)(\widehat{\delta}+m-1)\right)^2}.
\end{aligned}$$

It follows that $\frac{df(\widehat{\delta})}{d\widehat{\delta}} = 0$ when $\widehat{\delta} = m-1$. Moreover, for $\widehat{\delta} = m-1$, the second derivative of

$f(\widehat{\delta})$ evaluates to

$$\frac{d^2 f(\widehat{\delta})}{d\widehat{\delta}^2}\Big|_{\widehat{\delta}=m-1} = \frac{\left(\widehat{\delta}^2 + (m-1)(\widehat{\delta}+m-1)\right)^2 \left(-(m-1)2\widehat{\delta}\right)}{\left(\widehat{\delta}^2 + (m-1)(\widehat{\delta}+m-1)\right)^4}\Big|_{\widehat{\delta}=m-1}$$
$$= -\frac{2(m-1)\widehat{\delta}}{\left(\widehat{\delta}^2 + (m-1)(\widehat{\delta}+m-1)\right)^2}\Big|_{\widehat{\delta}=m-1}$$
$$< 0\,.$$

Thus, $\widehat{\delta} = m-1$ is a local maximum of the function $f(\widehat{\delta})$. Since $f(\widehat{\delta})$ is a continuous function with a single extremum point that is a local maximum, it follows that

$$f(\widehat{\delta}) \leq \frac{(m-1)^2}{(m-1)^2 + 2(m-1)^2} = \frac{1}{3}\,,$$

so that the value of the optimization problem Π_3 is $1+\frac{1}{3} = \frac{4}{3}$. It follows that $\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} \leq \frac{4}{3}$, as needed.

We continue to prove the lower bound. Fix an instance $\langle \mathbf{w},\mathbf{c}\rangle$ with $n = 2(m-1)$; for each link $l \in [m]$, set $c_l = 2(m-1)$ if $l = 1$ and 1 otherwise. Consider pure assignments $\mathbf{P}$ and $\mathbf{Q}$, such that:

- In $\mathbf{P}$, all users are assigned to link 1. Note that for each user $i \in [2(m-1)]$ $\lambda_i^1 = \frac{2(m-1)}{2(m-1)} = 1$ and for each link $l \in [m]$, $l \neq 1$, $\lambda_i^l = 1$; so, $\mathbf{P}$ is a Nash equilibrium. Clearly, $\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P}) = \frac{(2(m-1))^2}{2(m-1)} = 2(m-1)$.
- In $\mathbf{Q}$, $m-1$ users are assigned to link 1 and each of the remaining $m-1$ users is assigned to each link $2,\ldots,m$, respectively. Clearly, $\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q}) = \frac{(m-1)^2}{2(m-1)} + (m-1)\frac{1^2}{1} = \frac{3}{2}\,(m-1)$.

So, $\mathsf{QCR} \geq \frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} \geq \frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q})} = \frac{4}{3}$, as needed. ∎

5.1.2 Identical Links

We prove:

Theorem 5.5 *Consider the case of identical links, restricted to pure Nash equilibria. Then,* $\mathsf{QCR} = \frac{9}{8}$.

Proof: We first prove the upper bound. Consider any arbitrary instance $\langle \mathbf{w},\mathbf{c}\rangle$, with associated pure Nash equilibrium $\mathbf{P} = \langle \ell_1,\ldots,\ell_n\rangle$ and optimal assignment $\mathbf{Q} = \langle q_1,\ldots,q_n\rangle$. We distinguish between two cases:

1. <u>The instance $\langle \mathbf{w}, \mathbf{c} \rangle$ is non-bursty.</u> Recall that in this case, by Lemma 3.6, for each link $j \in [m]$, $\delta^j(\mathbf{P}) \leq 2 \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P})$. So transform the set of loads $\left\{ \delta^l(\mathbf{P}) \mid l \in [m] \right\}$ into a new set of loads $\left\{ \widehat{\delta}^l(\mathbf{P}) \mid l \in [m] \right\}$ as the output of the following repetitive procedure:

   > **for** each link $l \in [m]$, **do**
   >
   > $\quad\widehat{\delta}^l(\mathbf{P}) := \delta^l(\mathbf{P})$;
   >
   > **while** there are distinct $j, j' \in [m]$ with $\min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}) < \widehat{\delta}^j(\mathbf{P}) \leq \widehat{\delta}^{j'}(\mathbf{P}) < 2 \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P})$ **do**
   >
   > $$\begin{aligned} \widehat{\delta}^j(\mathbf{P}) &:= \widehat{\delta}^j(\mathbf{P}) - \min \left\{ \widehat{\delta}^j(\mathbf{P}) - \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}),\, 2 \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P}) \right\} \\ \widehat{\delta}^{j'}(\mathbf{P}) &:= \widehat{\delta}^{j'}(\mathbf{P}) + \min \left\{ \widehat{\delta}^j(\mathbf{P}) - \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}),\, 2 \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P}) \right\}. \end{aligned}$$
   >
   > **end while**

   Intuitively, our transformation procedure chooses at each step two intermediate loads $\delta^j(\mathbf{P})$ and $\delta^{j'}(\mathbf{P})$ (that is, two loads that are not yet pushed either to the upper end or to the lower end of the interval of link loads); it transfers the (strictly) positive quantity $\min \left\{ \widehat{\delta}^j(\mathbf{P}) - \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}),\, 2 \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P}) \right\}$ from the small load $\delta^j(\mathbf{P})$ to the large load $\delta^{j'}(\mathbf{P})$. Clearly, each step of the procedure either pushes the small load $\delta^j(\mathbf{P})$ to the lower end $\min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P})$ of the interval of link loads, or pushes the large load $\delta^{j'}(\mathbf{P})$ to the upper end $2 \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P})$ of the interval of link loads (or both). So, clearly, when the procedure terminates, there is at most one intermediate load. Hence, by reordering links, we obtain that for some integer $\widehat{j}$, $0 \leq \widehat{j} \leq m-1$, for each link $j \in [m]$,

   $$\widehat{\delta}^j(\mathbf{P}) = \begin{cases} 2 \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}), & j \in [\widehat{j}] \\ (1+x) \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}), & j = \widehat{j} + 1 \\ \min_{l \in [m]: \delta^l(\mathbf{P}) > 0} \delta^l(\mathbf{P}), & j \in [m] \setminus [\widehat{j} + 1] \end{cases}$$

   where $0 \leq x \leq 1$.

   Note that this transformation procedure maps a set of loads to a new set of loads, without explicitly mapping an instance to a new instance. However, for the sake of our analysis, we will also consider that the transformation procedure maps an instance $\langle \mathbf{w}, \mathbf{c} \rangle$ and a Nash equilibrium $\mathbf{P}$ to a new instance $\langle \widehat{\mathbf{w}}, \widehat{\mathbf{c}} \rangle$ and a new Nash equilibrium $\widehat{\mathbf{P}}$. Note also that this transformation preserves (at each of its steps) the sum of loads. Hence, it also preserves the total traffic, so that $W = \widehat{W}$.

Consider any individual step of our repetitive procedure transforming the set of loads $\left\{\delta^l(\mathbf{P}) \mid l \in [m]\right\}$ into a new set of loads $\left\{\widehat{\delta}^l(\mathbf{P}) \mid l \in [m]\right\}$. Then, clearly,

$$\begin{aligned}
& \mathsf{QSC}\left(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{P}}\right) - \mathsf{QSC}\left(\mathbf{w}, \mathbf{c}, \mathbf{P}\right) \\
= \; & \left(\widehat{\delta}^j(\mathbf{P}) - \min\left\{\widehat{\delta}^j(\mathbf{P}) - \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}), 2 \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2 \\
& + \left(\widehat{\delta}^{j'}(\mathbf{P}) + \min\left\{\widehat{\delta}^j(\mathbf{P}) - \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}), 2 \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2 \\
& - \left(\widehat{\delta}^j(\mathbf{P})\right)^2 - \left(\widehat{\delta}^{j'}(\mathbf{P})\right)^2 \\
= \; & 2 \min\left\{\widehat{\delta}^j(\mathbf{P}) - \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}), 2 \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P})\right\} \cdot \left(\widehat{\delta}^{j'}(\mathbf{P}) - \widehat{\delta}^l(\mathbf{P})\right) \\
& +2 \left(\min\left\{\widehat{\delta}^j(\mathbf{P}) - \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}), 2 \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2 \\
> \; & 0\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \;\le\; & \mathsf{QSC}(\widehat{\mathbf{w}}, \widehat{\mathbf{c}}, \widehat{\mathbf{P}}) \\
= \; & \sum_{l\in[m]} \left(\widehat{\delta}^l(\mathbf{P})\right)^2 \\
= \; & \left(4\widehat{j} + (1+x)^2 + (m - \widehat{j} - 1)\right) \cdot \\
& \left(\min\left\{\widehat{\delta}^j(\mathbf{P}) - \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}), 2 \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2 \\
= \; & \left(x^2 + 2x + m + 3\widehat{j}\right) \cdot \\
& \left(\min\left\{\widehat{\delta}^j(\mathbf{P}) - \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}), 2 \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2 .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathsf{QOPT}(\mathbf{w},\mathbf{c}) &\geq \frac{W^2}{m} \\
&= \frac{\widehat{W}^2}{m} \\
&= \frac{\left(\sum_{j\in[m]} \widehat{\delta}^j(\mathbf{P})\right)^2}{m} \\
&= \frac{\left(2\widehat{j} + (1+x) + (m-\widehat{j}-1)\right)^2}{m} \cdot \\
&\quad \left(\min\left\{\widehat{\delta}^j(\mathbf{P}) - \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}), 2\min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2 \\
&= \frac{\left(x+m+\widehat{j}\right)^2}{m} \cdot \\
&\quad \left(\min\left\{\widehat{\delta}^j(\mathbf{P}) - \min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}), 2\min_{l\in[m]:\delta^l(\mathbf{P})>0} \delta^l(\mathbf{P}) - \widehat{\delta}^{j'}(\mathbf{P})\right\}\right)^2 .
\end{aligned}$$

It follows that

$$\mathsf{QCR} \leq \frac{(x^2+2x+m+3\widehat{j})\, m}{(x+m+\widehat{j})^2}.$$

Define the real function

$$f(k) = \frac{(x^2+2x+m+3\widehat{j})m}{(x+m+\widehat{j})^2}$$

of a real variable $\widehat{j}$. (The quantity x is taken as a parameter, while m is a fixed constant.) Clearly, $\mathsf{QCR} \leq \sup_{\widehat{j}} f(\widehat{j})$. To maximize the function $f(\widehat{j})$, observe that the first and second derivatives of $f(\widehat{j})$ are

$$\frac{df(\widehat{j})}{d\widehat{j}} = \frac{m}{(x+m+\widehat{j})^3} \cdot \left(-2x^2 - x + m - 3\widehat{j}\right)$$

and

$$\frac{d^2 f(\widehat{j})}{d\widehat{j}^2} = m \cdot \left(\frac{-3(x+m+\widehat{j})^3 - (-2x^2-x+m-3\widehat{j})\cdot 3(x+m+\widehat{j})^2}{(x+m+k)^6}\right),$$

respectively. The only root of $\frac{df(\widehat{j})}{d\widehat{j}}$ is $\widehat{j}_0 = \frac{m-x-2x^2}{3}$. For $\widehat{j} = \widehat{j}_0$, the second derivative evaluates to

$$\begin{aligned}
\frac{d^2 f(\widehat{j})}{d\widehat{j}^2}\Big|_{\widehat{j}=\widehat{j}_0} &= \frac{-81m}{8(x+2m-x^2)^3} \\
&< 0.
\end{aligned}$$

Thus, $\widehat{j}_0$ is a local maximum of the function $f(\widehat{j})$. Since $f(\widehat{j})$ is a continuous function with a single extremum point that is a local maximum, it follows that

$$
\begin{array}{ll}
f(\widehat{j}) & \\
\leq & f(\widehat{j}_0) \\
= & \dfrac{\left(x^2+2x+m+3\cdot\frac{m-x-2x^2}{3}\right)m}{\left(x+m+\frac{m-x-2x^2}{3}\right)^2} \\
= & \dfrac{9m}{4(2m+x-x^2)} \\
\leq & \dfrac{9m}{8m} \qquad\qquad (\text{since } 0 \leq x \leq 1) \\
= & \dfrac{9}{8},
\end{array}
$$

as needed.

2. The instance $\langle \mathbf{w}, \mathbf{c} \rangle$ is bursty. Denote $\mathcal{U}$ the (non-empty) set of bursty users. Recall that, by Lemma 3.5, $\mathcal{U}$ induces sets of solo links $\mathcal{L}_{\mathbf{P}}$ and $\mathcal{L}_{\mathbf{Q}}$ for the Nash equilibrium $\mathbf{P}$ and the optimal assignment $\mathbf{Q}$, respectively, so that $|\mathcal{L}_{\mathbf{P}}| = |\mathcal{U}|$ and $|\mathcal{L}_{\mathbf{Q}}| = |\mathcal{U}|$. Since links are identical, we assume that $\mathcal{L}_{\mathbf{P}} = \mathcal{L}_{\mathbf{Q}} = \mathcal{L}$, with $|\mathcal{L}| \geq 1$. So,

$$
\begin{aligned}
\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) &= \sum_{l \in \mathcal{L}} \left(\delta^l(\mathbf{P})\right)^2 + \mathsf{QSC}\left(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{P} \setminus (\mathcal{U}, \mathcal{L})\right) \\
&= \sum_{i \in \mathcal{U}} w_i^2 + \mathsf{QSC}\left(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{P} \setminus (\mathcal{U}, \mathcal{L})\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) &= \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{Q}) \\
&= \sum_{l \in \mathcal{L}} \left(\delta^l(\mathbf{P})\right)^2 + \mathsf{QSC}\left(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{Q} \setminus (\mathcal{U}, \mathcal{L})\right) \\
&= \sum_{i \in \mathcal{U}} w_i^2 + \mathsf{QSC}\left(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{Q} \setminus (\mathcal{U}, \mathcal{L})\right).
\end{aligned}
$$

Note first that the assignment $\mathbf{P} \setminus (\mathcal{U}, \mathcal{L})$ is a Nash equilibrium for the instance $\langle \mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L} \rangle$. Moreover, since $\mathbf{Q}$ is an optimal assignment for the instance $\langle \mathbf{w}, \mathbf{c} \rangle$, it follows that $\mathbf{Q} \setminus (\mathcal{U}, \mathcal{L})$ is an optimal assignment for the instance $\langle \mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L} \rangle$, so that $\mathsf{QSC}\left(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}, \mathbf{Q} \setminus (\mathcal{U}, \mathcal{L})\right) = \mathsf{QOPT}\left(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}\right)$. Thus,

$$
\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) = \sum_{i \in \mathcal{U}} w_i^2 + \mathsf{QOPT}\left(\mathbf{w} \setminus \mathcal{U}, \mathbf{c} \setminus \mathcal{L}\right).
$$

It follows that

$$\begin{aligned}\frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QOPT}(\mathbf{w},\mathbf{c})} &= \frac{\sum_{i\in\mathcal{U}} w_i^2 + \mathsf{QSC}\left(\mathbf{w}\setminus\mathcal{U},\mathbf{c}\setminus\mathcal{L},\mathbf{P}\setminus(\mathcal{U},\mathcal{L})\right)}{\sum_{i\in\mathcal{U}} w_i^2 + \mathsf{QOPT}\left(\mathbf{w}\setminus\mathcal{U},\mathbf{c}\setminus\mathcal{L}\right)}\\ &\leq \frac{\mathsf{QSC}\left(\mathbf{w}\setminus\mathcal{U},\mathbf{c}\setminus\mathcal{L},\mathbf{P}\setminus(\mathcal{U},\mathcal{L})\right)}{\mathsf{QOPT}\left(\mathbf{w}\setminus\mathcal{U},\mathbf{c}\setminus\mathcal{L}\right)}.\end{aligned}$$

So consider the instance $\langle\mathbf{w}\setminus\mathcal{U},\mathbf{c}\setminus\mathcal{L}\rangle$ and the associated pure Nash equilibrium $\mathbf{P}\setminus(\mathcal{U},\mathcal{L})$. There are two possibilities depending on whether the instance $\langle\mathbf{w}\setminus\mathcal{U},\mathbf{c}\setminus\mathcal{L}\rangle$ is bursty or not.

- Assume first that the smaller instance $\langle\mathbf{w}\setminus\mathcal{U},\mathbf{c}\setminus\mathcal{L}\rangle$ is non-bursty. Then, we are reduced to the previous case of non-bursty instances, and the upper bound follows inductively.
- Assume now that the smaller instance $\langle\mathbf{w}\setminus\mathcal{U},\mathbf{c}\setminus\mathcal{L}\rangle$ is bursty. We repeatedly identify the set of bursty users for the smaller instance, and we reduce this smaller instance to an even smaller instance that may be bursty or non-bursty. This procedure eventually yields a non-bursty instance (even the trivial one with one user), and the claim for the original bursty instance follows inductively.

The proof of the upper bound is now complete. We continue to prove the lower bound. Fix $n = 8$ and $m = 3$. Set $w_i = 1$ for $1 \leq i \leq 2$ and $w_i = \frac{1}{3}$ otherwise. Observe that $\mathbf{P} = \langle 1,1,2,2,2,3,3,3\rangle$ is a pure Nash equilibrium with $\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P}) = 2^2 + 1^2 + 1^2 = 6$. Moreover, consider the pure assignment $\mathbf{Q} = \langle 1,2,1,2,3,3,3,3\rangle$ that achieves $\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q}) = 3\cdot\left(\frac{4}{3}\right)^2 = \frac{16}{3}$. Thus, $\mathsf{QCR} \geq \frac{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{QSC}(\mathbf{w},\mathbf{c},\mathbf{Q})} = \frac{6}{\frac{16}{3}} = \frac{9}{8}$, as needed. ■

5.2 Mixed Nash Equilibria

Throughout this section, we focus on the case of identical links. The case of the fully mixed Nash equilibrium, is treated in Section 5.2.1. The case of idenbtical users is considered in Section 5.2.2.

5.2.1 Identical Links and Fully Mixed Nash Equilibrium

We prove:

Theorem 5.6 *Consider the case of identical links. Then, restricted to the fully mixed Nash equilibrium,*

$$\mathsf{QCR} = 2 - \max\left\{\frac{1}{n},\frac{1}{m}\right\}.$$

Our proof will make use of a simple claim, which we first state and prove. Roughly speaking, this claim considers a pair of user traffics; it establishes that if the larger is increased by some amount and the smaller is decreased by the same amount, then the Social Cost of the fully mixed Nash equilibrium strictly increases.

Lemma 5.7 (Traffic Shifting Lemma) *Given a traffic vector* $\mathbf{w}$ *and a number* $x > 0$*, assume that for some indices* $k, l \in [m]$*,* $w_k \geq w_l$*. Construct a traffic vector* $\widehat{\mathbf{w}}$ *with*

$$\widehat{w_i} \quad = \quad \begin{cases} w_i + x, & \text{if } i = k \\ w_i - x, & \text{if } i = l \\ w_i, & \text{otherwise} \end{cases} \quad .$$

Then,

$$W_1 + \frac{2}{m} W_2 \quad < \quad \widehat{W_1} + \frac{2}{m} \widehat{W_2} \, .$$

Proof: Clearly,

$$\begin{aligned}
& \left(\widehat{W_1} + \frac{2}{m} \widehat{W_2} \right) - \left(W_1 + \frac{2}{m} W_2 \right) \\
= \; & \left(\widehat{W_1} - W_1 \right) + \frac{2}{m} \left(\widehat{W_2} - W_2 \right) \\
= \; & (w_k + x)^2 + (w_l - x)^2 - w_k^2 - w_l^2 \\
& + \frac{2}{m} \left(\sum_{i \in [n], i \neq k,l} (w_k + x) w_i + \sum_{i \in [n], i \neq k,l} (w_l - x) w_i + (w_k + x)(w_l - x) \right) \\
& - \frac{2}{m} \left(\sum_{i \in [n], i \neq k,l} w_k w_i + \sum_{i \in [n], i \neq k,l} w_l w_i + w_k w_l \right) \\
= \; & 2x w_k - 2x w_l + 2x^2 + \frac{2}{m} \left(x w_l - x w_k - x^2 \right) \\
= \; & 2x \left(w_k - w_l + x + \frac{1}{m} (w_l - w_k - x) \right) \\
= \; & 2x \, (w_k - w_l + x) \left(1 - \frac{1}{m} \right) \\
> \; & 0 \, ,
\end{aligned}$$

which proves the claim. ∎

We now continue with the proof of Theorem 5.6.

Proof: We first prove the upper bound. Consider any arbitrary instance $\langle \mathbf{w}, \mathbf{c} \rangle$, with an associated fully mixed Nash equilibrium $\mathbf{F}$. We proceed by case analysis on how n and m compare to each other.

1. Assume first that $n \leq m$. We will prove an upper bound of $2 - \frac{1}{n}$. Note that in this case all users are solo in an optimal assignment, so that $\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) = W_1$. So,

$$\begin{aligned}
&\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} \\
&= \frac{W_1 + \frac{2}{m} W_2}{W_1} && \text{(by Theorem 4.2)} \\
&\leq \frac{W_1 + \frac{2}{n} W_2}{W_1} && \text{(since } n \leq m\text{)} \\
&= 1 + \frac{2W_2}{nW_1} \\
&= 1 + \frac{W^2 - W_1}{nW_1} \\
&= 1 + \frac{W^2}{nW_1} - \frac{1}{n} \\
&\leq 2 - \frac{1}{n} && \text{(since } W_1 \geq \frac{W^2}{n}\text{)} \\
&= 2 - \max\left\{\frac{1}{n}, \frac{1}{m}\right\} ,
\end{aligned}$$

as needed.

2. Assume now that $n > m$. We will prove a strict upper bound of $2 - \frac{1}{m}$. Define $r \in \{0, \ldots, m-1\}$ to be the least index such that

$$w_{r+1} \leq \frac{W - \sum_{i \in [r]} w_i}{m - r} .$$

Thus, clearly, for all indices $l < r$,

$$w_{l+1} > \frac{W - \sum_{i \in [r]} w_i}{m - l} .$$

Note that such an r exists since the inequality holds vacuously for $r = m - 1$. Use r to define the quantity

$$\omega = \frac{W - \sum_{i \in [n]} w_i}{m - r} .$$

Since $w_1 \geq w_2 \geq \ldots \geq w_n$, it follows that for all users $i \in [n]$ with $i \geq r + 1$, $w_i \leq \omega$ as well.

We prove that each user $k \in [r]$ is solo in some optimal assignment $\mathbf{Q}$ for the instance $\langle \mathbf{w}, \mathbf{c} \rangle$. For the basis case, where $k = 1$, note that $w_1 > \frac{W}{m}$. So, user 1 is bursty. Now, Lemma 3.5 implies that user 1 is solo in any optimal assignment for the instance $\langle \mathbf{w}, \mathbf{c} \rangle$. Assume inductively that users $1, \ldots, k-1$ are solo in some optimal assignment $\mathbf{Q}$, with corresponding solo links $1, \ldots, k-1$, and consider user k and the smaller instance

$\langle \mathbf{w} \setminus [k-1], \mathbf{c} \setminus [k-1] \rangle$. Since $w_k > \frac{W - \sum_{i \in [k-1]} w_i}{m-(k-1)}$, it follows that user i is bursty for the instance $\langle \mathbf{w} \setminus [k-1], \mathbf{c} \setminus [k-1] \rangle$. Thus, Lemma 3.5 implies that user k is solo in any optimal assignment $\mathbf{Q}'$ for the smaller instance $\langle \mathbf{w} \setminus [k-1], \mathbf{c} \setminus [k-1] \rangle$, and the claim follows. Hence, it follows that

$$\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) = \sum_{i \in [r]} w_i^2 + \mathsf{QOPT}(\mathbf{w} \setminus [r], \mathbf{c} \setminus [r]) .$$

However,

$$\begin{aligned}
\min_{\mathbf{w} \setminus [r], \mathbf{c} \setminus [r]} \mathsf{QOPT}(\mathbf{w} \setminus [r], \mathbf{c} \setminus [r]) &= \min_{\sum_{l \in [m]} \delta^l(\mathbf{P}) = W - \sum_{i \in [r]} w_i} \sum_{l \in [m] \setminus [r]} \left(\delta^l(\mathbf{P}))^2 \right)^2 \\
&= (m-r) \left(\frac{W - \sum_{i \in [r]} w_i}{m-r} \right)^2 \\
&= (m-r)\, \omega^2 .
\end{aligned}$$

It follows that

$$\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) \geq \sum_{i \in [r]} w_i^2 + (m-r)\, \omega^2 .$$

Recall that for each user $i \in [n] \setminus [r]$, $w_i \leq \omega$. So, all traffics of users $i \in [n] \setminus [r]$ are in the interval $[0, \omega]$. Transform now the traffic vector $\mathbf{w}$ into the traffic vector $\widehat{\mathbf{w}}$ as the output of the following repetitive procedure:

while there are indices k, l with $r < k \leq m < l \leq n$ and $0 < w_l \leq w_k < \omega$ **do**

$$\widehat{w_i} := \begin{cases} w_i + \min\{w_l, \omega - w_k\}, & \text{if } i = k \\ w_i - \min\{w_l, \omega - w_k\}, & \text{if } i = l \\ w_i, & \text{otherwise} \end{cases} .$$

end while

Intuitively, our transformation procedure only affects users $i \in [n] \setminus [r]$. The procedure chooses at each step two intermediate traffics w_k and w_l (that is, two traffics that are not yet pushed either to the upper end ω or to the lower end 0 of the interval of traffics; it transfers the (strictly) positive quantity $\min\{w_l, \omega - w_k\}$ from the small traffic w_l to the large traffic w_k. Clearly, each step of the procedure either pushes the small traffic w_l down to 0, or pushes the large traffic w_k up to ω (or both). So, clearly, when the procedure terminates, there can be at most one intermediate traffic (since, otherwise, the procedure would not terminate). Since, however, $\sum_{i \in [n] \setminus [r]} \widehat{w_i} = \sum_{i \in [n] \setminus [r]} = (m-r)\omega$, it follows that there is then no intermediate traffic (since, otherwise, the sum $\sum_{i \in [m] \setminus [r]} \widehat{w}_i$ would not be a multiple of ω.) Hence, it holds that

$$\widehat{w}_i = \begin{cases} w_i, & \text{if } i \in [r] \\ \omega, & \text{if } i \in [m] \setminus [r] \\ 0 & \text{if } i \in [n] \setminus [m] \end{cases}$$

This implies that $\widehat{W}_1 = \sum_{i \in [r]} w_i^2 + (m-r)\omega^2$. Note also that this transformation preserves (at each step) the total traffic, so that $\widehat{W} = W$. So,

$$\begin{aligned}
&\frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} \\
\leq\ & \frac{W_1 + \frac{2}{m} W_2}{\sum_{i \in [r]} w_i^2 + (m-r)\alpha^2} \quad \text{(by Theorem 4.2)} \\
<\ & \frac{\widehat{W}_1 + \frac{2}{m} \widehat{W}_2}{\widehat{W}_1} \quad \text{(by Lemma 5.7)} \\
=\ & 1 + \frac{2}{m} \frac{\widehat{W}_2}{\widehat{W}_1}. \\
=\ & 1 + \frac{\widehat{W}^2 - \widehat{W}_1}{m \widehat{W}_1} \\
=\ & 1 + \frac{\widehat{W}^2}{m \widehat{W}_1} - \frac{1}{m}.
\end{aligned}$$

Note, however, that

$$\begin{aligned}
&\widehat{W}_1 \\
=\ & \sum_{i \in [m]} \widehat{w}_i^2 \quad \text{(since } \widehat{w}_i = 0 \text{ for } i \in [n] \setminus [m]\text{)} \\
\geq\ & \min_{\sum_{l \in [m]} \delta = \widehat{W}} \left(\sum_{l \in [m]} \delta_l^2 \right)^2 \\
=\ & \frac{\widehat{W}^2}{m}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathsf{QCR} &< 2 - \frac{1}{m} \\
&= 2 - \max\left\{\frac{1}{n}, \frac{1}{m}\right\},
\end{aligned}$$

as needed.

It follows that in all cases,

$$\mathsf{QCR} \leq 2 - \max\left\{\frac{1}{n}, \frac{1}{m}\right\},$$

and this completes the proof of the upper bound.

We continue to prove the lower bound. Fix $n = m$. Set $w_i = 1$ for all users $i \in [n]$. Then, clearly, $\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) = W_1 + \frac{2}{m} W_2 = m \cdot 1^2 + \frac{2}{m} \frac{m(m-1)}{2} \cdot 1 = 2m - 1$. Moreover, $\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) = m \cdot 1^2 = m$. Thus, $\mathsf{QCR} \geq \dfrac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} = 2 - \frac{1}{m}$, as needed. ■

Note that since $n \geq 2$ and $m \geq 2$, the tight bound of $2 - \max\left\{\frac{1}{n}, \frac{1}{m}\right\}$ is at least $\frac{3}{2}$ and strictly less than 2. An inspection of the proof of Theorem 5.6 reveals that for the case $n \leq m$, the bound $2 - \max\left\{\frac{1}{n}, \frac{1}{m}\right\}$ is still tight (since we considered an instance with $m = n$ for the proof of the lower bound $2 - \frac{1}{m}$). For the case $n > m$, a lower bound strictly less than but arbitrarily close to $2 - \frac{1}{m}$ follows by considering an instance with $n = m + 1$ and $w_i = 1$ for $i \in [m]$ and ε for $i = m + 1$, where $\varepsilon > 0$ is an arbitrarily small constant.

5.2.2 Identical Users and Identical Links

We prove:

Theorem 5.8 *Consider the case of identical users and identical links. Then,*

$$\mathsf{QCR} \leq 1 + \min\left\{\frac{m-1}{n}, \frac{n-1}{m}\right\}.$$

Proof: Consider any arbitrary instance $\langle \mathbf{w}, \mathbf{c} \rangle$ with an associated Nash equilibrium $\mathbf{P}$. Then,

$$\begin{aligned} & \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \\ \leq\ & \mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{F}) \quad \text{(by Theorem 4.8)} \\ =\ & \tfrac{n(n+m-1)}{m} \quad \text{(by Corollary 4.6)}. \end{aligned}$$

Since $\mathsf{QOPT}(\mathbf{w}, \mathbf{c}) \geq n \max\left\{1, \frac{n}{m}\right\}$, it follows that

$$\begin{aligned} \frac{\mathsf{QSC}(\mathbf{w}, \mathbf{c}, \mathbf{P})}{\mathsf{QOPT}(\mathbf{w}, \mathbf{c})} &\leq \frac{n+m-1}{m} \min\left\{1, \frac{m}{n}\right\} \\ &= (n+m-1) \min\left\{\frac{1}{n}, \frac{1}{m}\right\} \\ &= \min\left\{1 + \frac{m-1}{n}, 1 + \frac{n-1}{m}\right\} \\ &= 1 + \min\left\{\frac{m-1}{n}, \frac{n-1}{m}\right\}, \end{aligned}$$

as needed. ■

We observe that for all $m \geq 2$ and $n \geq 2$,

$$\begin{aligned}
& \min\left\{\frac{m-1}{n}, \frac{n-1}{m}\right\} + \max\left\{\frac{1}{n}, \frac{1}{m}\right\} \\
\leq \; & \min\left\{\frac{m-1}{\min\{n,m\}}, \frac{n-1}{\min\{n,m\}}\right\} + \max\left\{\frac{1}{\min\{n,m\}}, \frac{1}{\min\{n,m\}}\right\} \\
= \; & \min\left\{\frac{m}{\min\{n,m\}}, \frac{n}{\min\{n,m\}}\right\} - \frac{1}{\min\{n,m\}} + \frac{1}{\min\{n,m\}} \\
= \; & 1\,.
\end{aligned}$$

This implies that the upper bound in Theorem 5.8, for the case of identical users and identical links, not only applies to all (mixed) Nash equilibria but it is no worse than the tight bound in Theorem 5.6, for the case of identical links, which applies only to the fully mixed Nash equilibrium. So, intuitively, Theorems 5.6 and 5.8 together suggest that for the case of identical links, considering arbitrary users but restricting to the fully mixed Nash equilibrium may have a more severe influence on Quadratic Coordination Ratio than considering all mixed Nash equilibria but restricting to identical users.

6 Conclusion and Directions for Further Research

6.1 Summary

We have presented a new, potentially rich model for selfish routing over non-cooperative networks as an interesting hybrid variant of the two best studied models for selfish routing, namely the KP model [20] and the W model [30]. Within this model, we focused on certain interesting algorithmic, combinatorial, structural and optimality properties of Nash equilibria; we also introduced and studied Quadratic Social Cost and Quadratic Coordination Ratio as interesting variants of the well studied Social Cost and Coordination Ratio, respectively, from the KP model [20].

Most interestingly, we presented a collection of *tight* bounds on Price of Anarchy [28] for our model; these are the *first* known constant (or bounded by a constant) bounds (independent of the number of users and the number of links) for models with unsplittable traffics. Our proof techniques highlight several interesting transformations of instances (user traffics and link capacities) that do not decrease Quadratic Social Cost; moreover, these techniques highlight the formulation of Quadratic Coordination Ratio as the value of a non-linear maximization problem. We believe that the full potential of our proof techniques is yet to be explored.

6.2 Subsequent Work

Some subsequent work has touched issues similar to those addressed in this work.

- Gairing *et al.* [17] introduce and study a discrete routing game as yet another hybridization of the KP model and the W model. In their model, the latency on a link is determined by an arbitrary, non-decreasing, convex function; in turn, the latency defines the Expected Individual Cost for a user (in the same way as in the KP model). However, the Social Cost is taken to be the sum of the Expected Individual Costs (as in the W model). When users are identical and latency functions are linear, this discrete routing game reduces to the special case of identical users for the model studied in this paper. We note that the set of Nash equilibria for the model of Gairing *et al.* [17] (defined through arbitrary, convex latency functions) is (in general) different from the set of Nash equilibria for the model in this paper (which coincides with the set of Nash equilibria for the KP model.

  The following results in the work of Gairing *et al.* [17] are related to the model studied in this paper; some of them, although more general, become less tight than corresponding results in this paper for the special cases we considered.

  - Gairing *et al.* [17, Theorem 1] prove that the fully mixed Nash equilibrium (when it exists) is the worst-case Nash equilibrium for the case of identical users in their model. This extends our Theorem 4.8 to the case of links with arbitrary, convex latency functions.
  - For the case of identical users and identical links with latency functions $f(x) = x^d$, where $d \geq 1$, Gairing *et al.* [17, Theorem 5] prove a tight bound of B_{d+1}, the $(d+1)$-th *Bell number,* on Coordination Ratio (for their model). For $d = 1$, this implies an upper bound of $B_2 = 2$ on Quadratic Coordination Ratio for the case of identical users and identical links in our model, which applies to all (mixed) Nash equilibria. Theorem 5.8 in this work provides a slightly better upper bound.
  - For the case of identical users and arbitrary links with polynomial latency functions of maximum degree d, Gairing *et al.* [17, Corollary 1] prove an upper bound of $d+1$ on Coordination Ratio, restricted to pure Nash equilibria. For $d = 1$, this implies an upper bound of 2 on Quadratic Coordination Ratio for the case of identical users and identical links in our model, restricted to pure Nash equilibria. Theorem 5.1 provides a significantly better upper bound ($\frac{4}{3}$).

- Finally, Gairing *et al.* [15] introduce and study yet another hybridization of the KP model and the W model. In their model, the latency on a link is a linear function (as in both the KP model and the model studied in this paper). However, a so called *Polynomial Social Cost* is adopted that extends Quadratic Social Cost to arbitrary degrees. In turn, this yields Polynomial Coordination Ratio. Gairing *et al.* [15] only consider the case of identical users and identical links. For this case, they present a collection of upper bounds on Polynomial Coordination Ratio.

6.3 Open Problems

We conclude this article with a collection of questions that naturally pose themselves within the new model for selfish routing introduced in this work.

1. What is the time complexity of computing Quadratic Social Cost of an arbitrary Nash equilibrium in the new model? Proposition 4.1 provides a polynomial algorithm with time complexity $O(mn)$ for any arbitrary assignment. Is this optimal, or can it be further improved? (Lower bounds are totally missing.) Can this be improved when the assignement is a Nash equilibrium?

2. What is the time complexity of deciding optimality (with respect to Quadratic Social Cost) for a given pure assignment? Proposition 3.2 implies an $O(n + m^2)$ upper bound for the case of identical users. Is there a polynomial upper bound for the general case?

3. Prove the Quadratic Fully Mixed Nash Equilibrium Conjecture (Conjecture 4.1).

4. What is the value of Quadratic Coordination Ratio, when restricted to pure Nash equilibria, for the new model? Theorems 5.1 and 5.5 consider separately the two cases of identical links and identical users. It is a very challenging task to merge the two separate proofs for the two special cases into a proof for the general case (restricted to pure Nash equilibria).

5. What is the value of Quadratic Coordination Ratio when all Nash equilibria are considered? This will extend Theorems 5.6 and 5.8. In particular, no bound on Quadratic Coordination Ratio (for all Nash equilibria) is known for the general case of related links.

6. Extending the study of our hybrid model to arbitrary networks (following the W model) remains perhaps the most challenging direction for future research.

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